

# COMPOUND POISSON STATISTICS IN CONVENTIONAL AND NONCONVENTIONAL SETUPS

ARIEL RAPAPORT

**ABSTRACT.** Given a periodic point  $\omega$  in a  $\psi$ -mixing shift with countable alphabet, the sequence  $\{S_n\}$  of random variables counting the number of multiple returns to shrinking cylindrical neighborhoods of  $\omega$  is considered. Necessary and sufficient conditions for the convergence in distribution of  $\{S_n\}$  are obtained, and it is shown that the limit is a Pólya–Aeppli distribution. A global condition on the shift system, which guarantees the convergence in distribution of  $\{S_n\}$  for every periodic point, is introduced. This condition is used to derive results for  $f$ -expansions and Gibbs measures. Results are also obtained concerning the possible limit distribution of sub-sequences  $\{S_{n_k}\}$ . A family of examples in which there is no convergence is presented. We exhibit also an example for which the limit distribution is pure Poissonian.

## 1. INTRODUCTION

In this article  $(\Omega, T)$  is a shift space with a countable alphabet, equipped with a  $\psi$ -mixing and  $T$ -invariant probability measure  $\mathbb{P}$ . Given  $\omega \in \Omega$ , the sequence of random variables  $\{S_n^\omega\}_{n=1}^\infty$  is considered, where for each  $n \geq 1$ ,

$$S_n^\omega(\gamma) = \sum_{k=1}^{N_n^\omega} \prod_{j=1}^{\ell} 1_{A_n^\omega \circ T^{d_j k}}(\gamma) \quad \text{for } \gamma \in \Omega,$$

$A_n^\omega$  is the cylinder set determined by the  $n$ -prefix of  $\omega$ ,  $N_n^\omega = [(\mathbb{P}(A_n^\omega))^{-\ell}]$ , and  $1 \leq d_1 < \dots < d_\ell$  are fixed integers. When the setup is said to be conventional it means that  $\ell = 1$  and  $d_1 = 1$ . We are interested in the limit distribution  $\mu_\omega$ , of the sequence  $\{S_n^\omega\}$ .

In the conventional case, the existence and characterization of  $\mu_\omega$  is a widely studied problem. Poisson approximation estimates for almost all non-periodic  $\omega$ , were obtained in [AV1] and [AV2]. For a periodic  $\omega$ , compound Poisson approximations were derived in [HV]. In [FFT], it was shown that the distributional limit of the normalized number of returns to small neighborhoods of periodic points is

---

*Date:* April 14, 2014.

*2000 Mathematics Subject Classification.* Primary: 60F05 Secondary: 37D35, 60J05.

*Key words and phrases.*  $\psi$ -mixing, Pólya–Aeppli distribution, Gibbs measure,  $f$ -expansion.

Supported by ERC grant 306494.

compound Poisson. This was shown for certain non-uniformly hyperbolic dynamical systems, which include certain piecewise expanding maps of the interval.

The problem, in the nonconventional setup described above, was considered for the first time in [K]. It was shown there that if  $(\Omega, T)$  is a subshift of finite type and  $\mathbb{P}$  is a Gibbs invariant measure, then  $\mu_\omega$  exists (i.e.  $\{S_n^\omega\}$  has a limit in distribution) and is equal to the Poisson distribution with the parameter 1, for  $\mathbb{P}$ -almost all  $\omega \in \Omega$ . In [KR] this result was extended for every non-periodic  $\omega$  under more general  $\psi$ -mixing assumptions. It was also shown there that if  $\omega$  is a periodic point and the dynamical system is a Bernoulli shift with a countable state space, then  $\mu_\omega$  is a Pólya–Aeppli distribution.

Here the case of a periodic  $\omega$  is considered for a general  $\psi$ -mixing system in the nonconventional setup. It is shown that if  $\mu_\omega$  exists, then it must be a Pólya–Aeppli distribution (which may be pure Poissonian). A sufficient condition for this is given, which was first introduced for the conventional setup in [HV]. A necessary condition is also given which is the same as the sufficient condition for the conventional setup. Both the necessary condition and the fact that  $\mu_\omega$  must be a Pólya–Aeppli distribution (if it exists) are new also for the conventional case.

A verifiable global condition on the dynamical system that guarantees the existence of  $\mu_\omega$  for each periodic  $\omega$  is introduced. This condition is based on the notion of the inverse Jacobian of a measure preserving transformation, which is defined in the next section. Using this, it is shown that if the shift system is derived from an  $f$ -expansion on  $[0, 1]$  with its absolutely continuous invariant measure (see [H]) then  $\mu_\omega$  always exists, in particular this applies to the Gauss map equipped with the Gauss measure. For the conventional case this follows from results found in [FFT]. The global condition is also used for showing that if  $(\Omega, T)$  is of finite type and  $\mathbb{P}$  is a Gibbs measure then  $\mu_\omega$  always exists. This was first shown for the conventional setup in [HV].

If  $\omega$  is periodic then the sequence  $\{S_n^\omega\}$  does not necessarily have a limit in distribution, an example of this phenomena was given in [KR]. However, the partial limits in distribution of  $\{S_n^\omega\}$  can be characterized. It is shown here that if a sub-sequence  $\{S_{n_k}^\omega\}$  converges in distribution, then the limit distribution must be a compound Poisson distribution. Also, the nonconvergence example from [KR] is extended into a wide class of examples in the same spirit. In addition, an example of a finite type system and a periodic point  $\omega$  is given, for which  $\mu_\omega$  equals a pure Poisson distribution.

The rest of this article is organized as follows: In Section 2 the notation and framework being used are presented. In Section 3 the results are stated. In Section 4

we prove the results regarding the pointwise conditions. In Section 5 we prove the result concerning the global condition, and the two applications of it are obtained. In Section 6 the results regarding the classification of partial limits are proved. In Section 7 the family of nonconvergence examples is developed. In Section 8 we construct the example of a periodic point for which the limit distribution is pure Poissonian.

**Acknowledgment.** I would like to thank my adviser Professor Yuri Kifer, for suggesting to me problems studied in this paper, and for many helpful discussions.

## 2. FRAMEWORK AND NOTATIONS

**2.1. The underlying dynamical system.** Let  $\mathcal{A}$  be a finite or countable set (the alphabet), with  $|\mathcal{A}| > 1$ . Let  $(\Omega, \mathcal{F}, \mathbb{P}, T)$  be a measure-preserving system where  $\Omega = \mathcal{A}^{\mathbb{N}}$ ,  $\mathcal{F}$  is the  $\sigma$ -algebra generated by the coordinates projections from  $\Omega$  onto  $\mathcal{A}$ ,  $T: \Omega \rightarrow \Omega$  is the left shift, and  $\mathbb{P}$  is a  $T$ -invariant probability measure.  $\mathcal{A}$  is considered as a topological space with the discrete topology, and  $\mathcal{A}^{\mathbb{N}}$  as a topological space with the product topology. For distinct  $\omega, \gamma \in \Omega$  define  $d(\omega, \omega) = 0$  and  $d(\omega, \gamma) = 2^{-m}$ , where  $m = \min\{j \geq 0 : \omega_j \neq \gamma_j\}$ . Then  $d$  is a metric on  $\Omega$  which induces the product topology.

Given  $J \subset \mathbb{N}$  define

$$\mathcal{F}_J = \sigma\{\{\omega_j = a\} : j \in J \text{ and } a \in \mathcal{A}\}$$

We assume that  $\mathbb{P}$  is  $\psi$ -mixing, i.e. that there exists a sequence  $\{\psi_m\}_{m \geq 0} \subset \mathbb{R}^+$  with  $\psi_m \xrightarrow{m \rightarrow \infty} 0$ , such that for each  $m, n \in \mathbb{N}$ ,  $E \in \mathcal{F}_{\{0, \dots, n-1\}}$  and  $F \in \mathcal{F}$ ,

$$(2.1) \quad |\mathbb{P}(E \cap T^{-(n+m)}F) - \mathbb{P}(E)\mathbb{P}(F)| \leq \psi_m \mathbb{P}(E)\mathbb{P}(F)$$

Examples of  $T$ -invariant and  $\psi$ -mixing measures include Gibbs measures on a subshift of finite type (see [B2]), and measures derived from  $f$ -expansions defined on the unit interval (see [H]).

Sometimes it will be more convenient to work in a certain closed subspace of  $\Omega$ , namely in a topological Markov shift (see [S]), which will now be defined. Let  $S = (S_{a,b})_{a,b \in \mathcal{A}}$  be a matrix of 0's and 1's with no columns or rows which are all 0's. Let

$$\Omega_S = \{\omega \in \Omega : S_{\omega_j, \omega_{j+1}} = 1 \text{ for each } j \geq 0\}$$

and

$$\mathcal{F}_S = \{E \in \mathcal{F} : E \subset \Omega_S\}$$

Then  $\Omega_S$  is a closed and  $T$ -invariant subset of  $\Omega$ , which is considered as a topological space with the subspace topology inherited from  $\Omega$ . It is assumed that  $\mathbb{P}(\Omega_S) = 1$ ,

and that  $T : \Omega_S \rightarrow \Omega_S$  is topologically mixing, i.e. for every pair of open sets  $U, V \subset \Omega_S$  there exist a number  $N(U, V) \in \mathbb{N}_+$  such that  $U \cap T^{-n}V \neq \emptyset$  for all  $n \geq N(U, V)$ .

**2.2. Words and cylinders.** Let  $\mathcal{A}^*$  be the set of finite words over  $\mathcal{A}$ , and let  $\mathcal{A}_S^* \subset \mathcal{A}^*$  be the subset of  $S$ -admissible words, i.e.

$$\mathcal{A}_S^* = \{a_0 \cdot \dots \cdot a_{r-1} \in \mathcal{A}^* : S_{a_{j-1}, a_j} = 1 \text{ for each } 1 \leq j < r\}$$

For each  $u, w \in \mathcal{A}^*$ ,  $\omega \in \Omega$  and  $k \geq 0$ , let  $u \cdot w \in \mathcal{A}^*$  be the concatenation of  $u$  and  $w$ , let  $w^k \in \mathcal{A}^*$  be the concatenation of  $w$  with itself  $k$  times, and let  $u \cdot \omega \in \Omega$  be the sequence obtained by adding  $u$  to the beginning of  $\omega$ . Given  $a_0, \dots, a_{r-1} \in \mathcal{A}$ ,  $a_0 \cdot \dots \cdot a_{r-1} = w \in \mathcal{A}^*$  and  $n \geq 1$  let

$$w^{n/r} = w^{[n/r]} \cdot a_0 \cdot \dots \cdot a_{n-r[\frac{n}{r}]-1}$$

where  $[\frac{n}{r}]$  stands for the integral part of  $\frac{n}{r}$ , and let

$$[w] = \{\omega \in \Omega : \omega_j = a_j \text{ for each } 0 \leq j < r\}.$$

The set  $[w]$  is called an  $r$ -cylinder. Sometimes  $[w]^{n/r}$  is written in place of  $[w^{n/r}]$ , and  $[a_0, \dots, a_{r-1}]$  in place of  $[w]$ .

From the  $\psi$ -mixing assumption and Lemma 3.1 in [KR] it follows that there exist a constant  $\Gamma > 0$  such that

$$(2.2) \quad \mathbb{P}[a_0, \dots, a_{n-1}] \leq e^{-\Gamma n}$$

for each  $a_0, \dots, a_{n-1} \in \mathcal{A}$ .

Given an  $r$ -cylinder  $A$  let  $\pi(A)$  be the period of  $A$ , i.e.

$$\pi(A) = \min\{j \in \{1, \dots, r\} : A \cap T^{-j}A \neq \emptyset\}$$

Given  $\omega \in \Omega$  and  $n \geq 1$ , define  $A_n^\omega = [\omega_0, \dots, \omega_{n-1}]$ . Let  $\Omega_{\mathbb{P}} \subset \Omega_S$  be the support of  $\mathbb{P}$ , then

$$\Omega_{\mathbb{P}} = \{\omega \in \Omega : \mathbb{P}(A_n^\omega) > 0 \text{ for each } n \geq 1\}$$

**2.3. The observables.** The random variables counting the number of multiple recurrences to cylindrical neighborhoods will now be defined. Let  $1 \leq d_1 < \dots < d_\ell$  be integers. For each cylinder  $A \in \mathcal{F}$ ,  $N \in \mathbb{N}_+ := \{1, 2, \dots\}$  and  $\omega \in \Omega$  set

$$S_N^A(\omega) = \sum_{k=1}^N X_k^A(\omega) \text{ where } X_k^A(\omega) = \prod_{j=1}^{\ell} 1_A \circ T^{d_j k}(\omega) \text{ for each } k \in \mathbb{N}_+$$

where  $1_A$  stands for the indicator function of the set  $A$ . As mentioned above, when we say that the setup is conventional it means that  $\ell = 1$  and  $d_1 = 1$ .

**2.4. The inverse Jacobian.** For each measurable  $E \in \mathcal{F}_S$  define

$$\mathbb{P} \circ T(E) = \sum_{a \in \mathcal{A}} \mathbb{P}(T(E \cap [a]))$$

then  $\mathbb{P} \circ T$  is a  $\sigma$ -finite measure on  $(\Omega_S, \mathcal{F}_S)$ , which is finite on cylinders (see [S] for more details on  $\mathbb{P} \circ T$ ). Given  $E \in \mathcal{F}_S$ ,

$$\mathbb{P} \circ T(E) = \sum_{a \in \mathcal{A}} \mathbb{P}(T^{-1}(T(E \cap [a]))) \geq \sum_{a \in \mathcal{A}} \mathbb{P}(E \cap [a]) = \mathbb{P}(E),$$

so  $\mathbb{P} \ll \mathbb{P} \circ T$ , and the following definition makes sense.

**Definition.** The function  $J = \frac{d\mathbb{P}}{d\mathbb{P} \circ T} \in L^1(\Omega_S, \mathcal{F}_S, \mathbb{P} \circ T)$  will be called here the inverse Jacobian of  $\mathbb{P}$ .

The global condition mentioned in the introduction involves the concept of the inverse Jacobian. More on this notion can be found in [S] and [W]. In [S] the function  $\frac{d\mathbb{P}}{d\mathbb{P} \circ T}$  is called the Jacobian, and in [W] this name is given to  $\frac{d\mathbb{P} \circ T}{d\mathbb{P}}$ . Here we use the function  $\frac{d\mathbb{P}}{d\mathbb{P} \circ T}$ , since always  $\mathbb{P} \ll \mathbb{P} \circ T$  whenever  $\mathbb{P}$  is  $T$ -invariant.

**2.5. Probability measures on  $\mathbb{N}$ .** Let  $\mathcal{M}(\mathbb{N})$  denote the collection of all probability measures on  $\mathbb{N}$ . Given a random variable  $Y$ , the distribution of  $Y$  is denoted by  $\mathcal{L}(Y)$ . Given  $\mu \in \mathcal{M}(\mathbb{N})$ , it is written  $Y \sim \mu$  if  $Y$  is a random variable with  $\mathcal{L}(Y) = \mu$ . Given random variables  $Y, Y_1, Y_2, \dots$  we write  $Y_n \xrightarrow{d} Y$  if the sequence  $\{Y_j\}_{j=1}^\infty$  converges to  $Y$  in distribution.

The total variation distance between members of  $\mathcal{M}(\mathbb{N})$  is denoted by  $d_{TV}$ , i.e. given  $\mu, \nu \in \mathcal{M}(\mathbb{N})$ ,

$$d_{TV}(\mu, \nu) = \sup\{|\mu(E) - \nu(E)| : E \subset \mathbb{N}\}$$

Given  $\mu, \mu_1, \mu_2, \dots \in \mathcal{M}(\mathbb{N})$  we write  $\mu_j \xrightarrow{d} \mu$  if the sequence  $\{\mu_j\}_{j=1}^\infty$  converges to  $\mu$  in distribution. Then  $\mu_j \xrightarrow{d} \mu$  if and only if  $d_{TV}(\mu_j, \mu) \xrightarrow{j} 0$ , which holds if and only if  $|\mu\{k\} - \mu_j\{k\}| \xrightarrow{j} 0$  for each  $k \in \mathbb{N}$ . See [BHJ] for more details on the total variation distance.

A sequence  $\{\mu_j\}_{j=1}^\infty \subset \mathcal{M}(\mathbb{N})$  is said to be tight if for every  $\epsilon > 0$  there exists  $N \geq 1$  such that  $\mu_j[N, \infty) \leq \epsilon$  for each  $j \geq 1$ . It holds that  $\{\mu_j\}_{j=1}^\infty$  is tight if and only if for every sub-sequence  $\{\mu_{j_k}\}_{k=1}^\infty$  there exist a further sub-sequence  $\{\mu_{j_{k_i}}\}_{i=1}^\infty$  and  $\mu \in \mathcal{M}(\mathbb{N})$ , such that  $\mu_{j_{k_i}} \xrightarrow{d} \mu$  as  $i \rightarrow \infty$  (see Theorem 25.10 in [B1]).

Given a random variable  $Y$ , the characteristic function of  $Y$  is denoted by  $\varphi_Y$ , i.e.  $\varphi_Y(x) = E[e^{ixY}]$  for each  $x \in \mathbb{R}$ . Given  $\mu \in \mathcal{M}(\mathbb{N})$ , the characteristic function of  $\mu$  is denoted by  $\varphi_\mu$ , i.e.  $\varphi_\mu(x) = \int e^{ixy} d\mu(y)$  for each  $x \in \mathbb{R}$ .

The following members of  $\mathcal{M}(\mathbb{N})$  will appear later on.

2.5.1. *The Poisson distribution.* For  $0 < t \in \mathbb{R}$  denote by  $Pois(t) \in \mathcal{M}(\mathbb{N})$  the Poisson distribution with parameter  $t$ , which satisfies

$$Pois(t)\{k\} = e^{-t} \frac{t^k}{k!}$$

for each  $k \in \mathbb{N}$ .

2.5.2. *The Geometric distribution.* For  $p \in [0, 1)$  denote by  $Geo(p) \in \mathcal{M}(\mathbb{N})$  the geometric distribution with success parameter  $p$ , which satisfies

$$Geo(p)\{k\} = (1 - p)p^{k-1}$$

for each  $k \in \mathbb{N}_+$ .

2.5.3. *The Compound Poisson distribution.* For  $0 < t \in \mathbb{R}$  and  $\nu \in \mathcal{M}(\mathbb{N})$ , denote by  $CP(t, \nu) \in \mathcal{M}(\mathbb{N})$  the compound Poisson distribution with parameters  $t$  and  $\nu$ , which satisfies

$$CP(t, \nu)\{k\} = \sum_{j=1}^{\infty} Pois(t)\{j\} \cdot (\nu*)^j\{k\}$$

for each  $k \in \mathbb{N}$ , where  $(\nu*)^j$  is the  $j$ -fold convolution of  $\nu$ . Let  $W \sim Pois(t)$ , and let  $\eta_1, \eta_2, \dots$  be i.i.d random variables independent of  $W$  with  $\eta_1 \sim \nu$ , then  $\sum_{j=1}^W \eta_j \sim CP(t, \nu)$ . Also, one checks that

$$(2.3) \quad \varphi_{CP(t, \nu)}(x) = \exp(t(\varphi_{\nu}(x) - 1))$$

for each  $x \in \mathbb{R}$ .

2.5.4. *The Pólya–Aeppli distribution.* For  $0 < t \in \mathbb{R}$  and  $p \in [0, 1)$ , denote by  $PA(t, p) \in \mathcal{M}(\mathbb{N})$  the Pólya–Aeppli distribution with parameters  $t$  and  $p$ , which satisfies

$$PA(t, p)\{k\} = e^{-t} \sum_{j=1}^k \binom{k-1}{j-1} \frac{t^j}{j!} p^{k-j} (1-p)^j$$

for each  $k \in \mathbb{N}_+$ , and  $PA(t, p)\{0\} = e^{-t}$ . One checks that  $PA(t, p) = CP(t, Geo(p))$ , and so from (2.3),

$$(2.4) \quad \varphi_{PA(t, p)}(x) = \exp(t(\varphi_{Geo(p)}(x) - 1))$$

for each  $x \in \mathbb{R}$ . Observe that  $PA(t, 0) = Pois(t)$ .

### 3. STATEMENT OF THE RESULTS

3.1. **Pointwise conditions for convergence and nonconvergence.** Throughout this article  $t > 0$  will be a fixed parameter. For each  $\omega \in \Omega_{\mathbb{P}}$  and  $n \geq 1$ , let

$N_n^\omega = [t(\mathbb{P}(A_n^\omega))^{-\ell}]$ . For each  $r \in \mathbb{N}_+$  set

$$\kappa(r) = \text{lcm}\left\{\frac{r}{\gcd\{r, d_j\}} : 1 \leq j \leq \ell\right\}$$

where  $\text{lcm}$  and  $\gcd$  denote the least common multiple and the greatest common divisor, respectively. For an  $n$ -cylinder  $A = [a_0, \dots, a_{n-1}]$  with  $\mathbb{P}(A) > 0$ ,  $r = \pi(A)$  and  $R = [a_0, \dots, a_{r-1}]$ , set

$$\rho_A = \prod_{j=1}^{\ell} \mathbb{P}\{R^{(n+d_j\kappa(r))/r} \mid A\}.$$

From (2.12) of Theorem 2.3 in [KR], it follows that

$$(3.1) \quad \sup\{\rho_A : n \geq 1, A \text{ is an } n\text{-cylinder}, \mathbb{P}(A) > 0\} < 1$$

For a periodic point  $\omega \in \Omega_{\mathbb{P}}$  with minimal period  $r \geq 1$  (i.e.

$$r = \inf\{j \geq 1 : T^j\omega = \omega\}$$

and  $r < \infty$ ) define  $\beta_{\omega,n} = \mathbb{P}\{A_{n+r}^\omega \mid A_n^\omega\}$  for each  $n \geq 1$ . The following simple lemma will be proven in Section 4.

**Lemma 1.** *Let  $\omega \in \Omega_{\mathbb{P}}$  be a periodic point with minimal period  $r \geq 1$ . Assume that the limit  $\beta_\omega = \lim_{n \rightarrow \infty} \beta_{\omega,n}$  exists. Then the limit  $\rho_\omega = \lim_{n \rightarrow \infty} \rho_{A_n^\omega}$  exists, and is equal to  $\beta_\omega^a$ , where  $a = \frac{\kappa(r)}{r} \sum_{i=1}^{\ell} d_i$ . Also, it holds that  $\rho_\omega < 1$ .*

The pointwise conditions can now be stated.

**Theorem 2.** *Let  $\omega \in \Omega_{\mathbb{P}}$  be a periodic point with a minimal period  $r \geq 1$ , then:*

(a) *If  $\lim_{n \rightarrow \infty} \rho_{A_n^\omega}$  does not exist then  $\{S_{N_n^\omega}^{A_n^\omega}\}_{n=1}^\infty$  does not converge in distribution.*

(b) *If the limit  $\beta_\omega = \lim_{n \rightarrow \infty} \beta_{\omega,n}$  exists then  $\mathcal{L}(S_{N_n^\omega}^{A_n^\omega}) \xrightarrow{d} PA(t(1-\rho_\omega), \rho_\omega)$  as  $n \rightarrow \infty$ .*

*Remark.* Assertion (b) was first proven for the conventional setup in [HV].

*Remark.* Note that if  $\rho_\omega = 0$  then  $\mathcal{L}(S_{N_n^\omega}^{A_n^\omega}) \xrightarrow{d} \text{Pois}(t)$ . In Section 3.5 an example in which this situation occurs will be presented.

*Remark.* Observe that in the conventional setup Theorem 2 says that  $\beta_\omega = \lim_{n \rightarrow \infty} \beta_{\omega,n}$  exists if and only if  $\{S_{N_n^\omega}^{A_n^\omega}\}_{n=1}^\infty$  converges in distribution, in which case

$$\mathcal{L}(S_{N_n^\omega}^{A_n^\omega}) \xrightarrow{d} PA(t(1-\beta_\omega), \beta_\omega) \text{ as } n \rightarrow \infty.$$

### 3.2. A global condition for convergence and applications.

3.2.1. *Continuous inverse Jacobian.* A condition on the dynamical system will now be stated that guarantees the convergence in distribution of  $\{S_{N_n^\omega}^{A_n^\omega}\}_{n=1}^\infty$ , for every periodic point  $\omega \in \Omega_{\mathbb{P}}$ .

**Theorem 3.** *Assume that the inverse Jacobian  $J = \frac{d\mathbb{P}}{d\mathbb{P} \circ T}$  is a continuous function on  $\Omega_S$  (i.e. it has a continuous version), where  $\Omega_S$  and its topology were defined in Section 2.1. Let  $\omega \in \Omega_{\mathbb{P}}$  be a periodic point with minimal period  $r \geq 1$ , then the limit  $\lim_{n \rightarrow \infty} \beta_{\omega, n}$  exists and it is equal to  $\prod_{j=0}^{r-1} J(T^j \omega)$ .*

The next corollary follows immediately from Theorems 2 and 3.

**Corollary 4.** *Assume that the inverse Jacobian  $J = \frac{d\mathbb{P}}{d\mathbb{P} \circ T}$  is a continuous function on  $\Omega_S$ . Let  $\omega \in \Omega_{\mathbb{P}}$  be a periodic point with minimal period  $r \geq 1$ , then*

$$\mathcal{L}(S_{N_n^\omega}^{A_n^\omega}) \xrightarrow{d} PA(t(1 - \rho_\omega), \rho_\omega) \text{ as } n \rightarrow \infty$$

where  $\rho_\omega = \left( \prod_{j=0}^{r-1} J(T^j \omega) \right)^a$  and  $a = \frac{\kappa(r)}{r} \sum_{i=1}^\ell d_i$ .

Two applications of Corollary 4 will now be stated.

3.2.2. *Gibbs measures.* In this section it will be assumed that the shift space is of finite type, i.e. that  $|\mathcal{A}| < \infty$ . Let  $\phi : \Omega_S \rightarrow \mathbb{R}$  be Hölder continuous with respect to the metric  $d$ , where  $d$  was defined in Section 2.1. From Theorem 1.4 in [B2] it follows that there exist a unique  $T$ -invariant Borel probability measure  $\mathbb{P}_\phi$  on  $\Omega_S$  for which one can find constants  $c_1 > 0$ ,  $c_2 > 0$  and  $P \in \mathbb{R}$  such that

$$(3.2) \quad c_1 \leq \frac{\mathbb{P}_\phi[\omega_0, \dots, \omega_{n-1}]}{\exp(-Pn + \sum_{j=0}^{n-1} \phi(T^j \omega))} \leq c_2$$

for each  $\omega \in \Omega_S$  and  $n \geq 0$ . The measure  $\mathbb{P}_\phi$  is called the Gibbs measure of  $\phi$  and  $P$  is called the pressure of  $\phi$ . It is shown in [B2] that  $\mathbb{P}_\phi$  is  $\psi$ -mixing. For each  $E \in \mathcal{F}$  define  $\mathbb{P}(E) = \mathbb{P}_\phi(E \cap \Omega_S)$ , then  $\mathbb{P}$  is as described in Section 2.1. Observe that from (3.2) it follows that  $\Omega_{\mathbb{P}} = \Omega_S$ .

**Theorem 5.** *Let  $J = \frac{d\mathbb{P}}{d\mathbb{P} \circ T}$  be the inverse Jacobian of  $\mathbb{P}$ . Then there exist a continuous function  $h : \Omega_S \rightarrow \mathbb{R}$  with  $h > 0$ , such that*

$$J(\omega) = e^{\phi(\omega) - P} \frac{h(\omega)}{h(T\omega)}$$

for  $\mathbb{P} \circ T$ -almost all  $\omega \in \Omega_S$ .

From Theorem 5 and Corollary 4 it follows that:

**Corollary 6.** *Let  $\omega \in \Omega_S$  be a periodic point with a minimal period  $r \geq 1$ , then*

$$\mathcal{L}(S_{N_n^\omega}^{A_n^\omega}) \xrightarrow{d} PA(t(1 - \rho_\omega), \rho_\omega) \text{ as } n \rightarrow \infty$$



where  $\rho_\omega = \exp(a \cdot \sum_{j=0}^{r-1} (\phi(T^j \omega) - P))$  and  $a$  is as in Corollary 4.

*Remark.* For the conventional case Corollary 6 was proven in [HV]. When  $\mathbb{P}$  is a Bernoulli shift Corollary 6 was proven in [KR] even for a countable alphabet.

*Remark.* Corollary 6 actually still holds for some Gibbs measures over a countable alphabet. In this case some additional assumptions should be made in order to insure the  $\psi$ -mixing property.

**3.2.3. Expanding Markov Interval Maps.** Another family of systems, for which Corollary 4 can be applied, will now be described. These systems are derived from certain expanding Markov interval maps. In [H] a family of dynamical systems is defined, namely the family of  $f$ -expansions with their absolutely continuous invariant measures, whose members satisfy all of the assumptions listed below. An example of such a system is the Gauss map,  $x \rightarrow \frac{1}{x} \pmod{1}$  for  $x \in (0, 1]$ , equipped with the Gauss measure  $\mu_G(\Gamma) = \frac{1}{\ln 2} \int_\Gamma \frac{1}{1+x} dx$ . If we use the measure  $\mu_G$  then the results from [KR] could not be applied to numbers having periodic continued fraction expansions (for instance the fractional parts of the golden ratio  $\frac{1+\sqrt{5}}{2}$  or  $\sqrt{2}$ ), while Corollary 8 below yields the Pólya–Aeppli limiting distribution in this case.

Let  $I = [0, 1]$  and let  $m$  denote the Lebesgue measure on  $I$ . Let  $\{U_a \subset I : a \in \mathcal{A}\}$  be a collection of disjoint open intervals. For each  $a \in \mathcal{A}$  set  $I_a = \overline{U_a}$ , set  $U = \bigcup_{a \in \mathcal{A}} U_a$  and assume that  $m(I \setminus U) = 0$ . Let  $f : I \rightarrow I$  be such that:

- (i) For every  $a \in \mathcal{A}$  the restriction of  $f$  to  $U_a$ , can be extended to a function  $f_a \in C^1(I_a)$  which is strictly monotonic.
- (ii) For every  $a, b \in \mathcal{A}$ , if  $f(U_a) \cap U_b \neq \emptyset$  then  $U_b \subset f(U_a)$ .
- (iii) For some integer  $q \geq 1$  there is a  $\tau > 1$ , such that  $|(f^q)'(x)| \geq \tau$  for all  $x \in \bigcap_{j=0}^{q-1} f^{-j}(U)$ .
- (iv) For every  $a \in \mathcal{A}$  there are  $b, c \in \mathcal{A}$  such that  $U_b \subset f(U_a)$  and  $U_a \subset f(U_c)$ .

Let  $\mu$  be an  $f$ -invariant Borel probability measure on  $I$  such that  $\mu \ll m$ . Set  $p = \frac{d\mu}{dm}$ , and assume that  $p$  is continuous and strictly positive.

Set  $N = \bigcup_{j=0}^{\infty} f^{-j}(I \setminus U)$  and  $\tilde{I} = I \setminus N$ . The set  $N$  is  $\mu$ -negligible, since

$$\mu(N) \leq \sum_{j=0}^{\infty} \mu(f^{-j}(I \setminus U)) = \sum_{j=0}^{\infty} \mu(I \setminus U) = \sum_{j=0}^{\infty} \int_{I \setminus U} p \, dm = 0$$

For each  $j \geq 0$ , let  $\xi_j : \tilde{I} \rightarrow \mathcal{A}$  be such that

$$\xi_j(x) = a \text{ if and only if } f^j(x) \in U_a$$

for each  $x \in \tilde{I}$  and  $a \in \mathcal{A}$ . It is assumed that the process  $\{\xi_j\}_{j=0}^\infty$  is  $\psi$ -mixing (see (1.8) in [H]).

Define the matrix  $S$ , such that  $S_{a,b} = 1$  if and only if  $U_b \subset f(U_a)$ , for each  $a, b \in \mathcal{A}$ . We assumed that  $T : \Omega_S \rightarrow \Omega_S$  is topologically mixing. It is not difficult to show (see Proposition 1.2 in [S]) that there exist a Hölder continuous map  $\Theta : \Omega_S \rightarrow I$  such that:

- (1) for every  $x \in \tilde{I}$  there is a unique  $\omega \in \Omega_S$  with  $\Theta(\omega) = x$ , and this  $\omega$  satisfies  $\omega_j = \xi_j(x)$  for each  $j \geq 1$ .
- (2)  $\Theta(\omega) \in I_{\omega_0}$  for each  $\omega \in \Omega_S$ .
- (3) if  $\omega \in \Omega_S$  and  $\Theta(\omega) \in \tilde{I}$  then  $\Theta(T\omega) = f(\Theta(\omega))$ .

For each  $\omega \in \Omega_S$ ,  $\Theta(\omega)$  is defined in [S] to be the unique element in  $\bigcap_{n \geq 1} \overline{\bigcap_{j=0}^{n-1} f^{-j}(U_{\omega_j})}$ .

Let  $\Phi : \tilde{I} \rightarrow \Omega$  be such that

$$\Phi(x) = (\xi_0(x), \xi_1(x), \dots)$$

for each  $x \in \tilde{I}$ . Then  $\Phi$  is the inverse of  $\Theta$  when  $\Theta$  is restricted to  $\Theta^{-1}(\tilde{I})$ . Let  $\mathbb{P} = \mu \circ \Phi^{-1}$  then, since  $\mu$  is  $f$ -invariant and  $\{\xi_j\}_{j=0}^\infty$  is  $\psi$ -mixing, it follows that  $\mathbb{P}$  is  $T$ -invariant and  $\psi$ -mixing.

**Theorem 7.** *Let  $J = \frac{d\mathbb{P}}{d\mathbb{P} \circ T}$  be the inverse Jacobian of  $\mathbb{P}$ , then  $J(\omega) = \frac{p(\Theta(\omega))}{f'_{\omega_0}(\Theta(\omega))p(\Theta(T\omega))}$  for  $\mathbb{P} \circ T$ -almost all  $\omega \in \Omega_S$ .*

From Theorem 7 and from Corollary 4 it follows that:

**Corollary 8.** *Let  $\omega \in \Omega_{\mathbb{P}}$  be a periodic point with minimal period  $r \geq 1$ , then*

$$\mathcal{L}(S_{N_n^\omega}^{A_n^\omega}) \xrightarrow{d} PA(t(1 - \rho_\omega), \rho_\omega) \text{ as } n \rightarrow \infty$$

where  $\rho_\omega = \left( \prod_{j=0}^{r-1} f'_{\omega_j}(\Theta(T^j\omega)) \right)^{-a}$ , and  $a$  is as in Corollary 4.

**3.3. Analysis of partial limits.** As we can see from the examples constructed in Section 3.4, for a periodic point  $\omega$  the sequence  $\{S_{N_n^\omega}^{A_n^\omega}\}$  does not necessarily has a limit in distribution. But its partial limits are always Compound Poisson.

**Theorem 9.** *Let  $\omega \in \Omega_{\mathbb{P}}$  be a periodic point, and for each  $n \geq 1$  let  $\mu_n \in \mathcal{M}(\mathbb{N})$  be the distribution of  $S_{N_n^\omega}^{A_n^\omega}$ . Then:*

- (a) *The sequence  $\{\mu_n\}$  is tight.*
- (b) *Let  $\mu$  be a probability distribution on  $\mathbb{R}$  such that  $\mu_{n_k} \xrightarrow{d} \mu$  as  $k \rightarrow \infty$ , for some sub-sequence  $\{\mu_{n_k}\}$ . Then  $\mu = CP(\tau, \theta)$ , for some  $0 < \tau \leq t$  and  $\theta \in \mathcal{M}(\mathbb{N})$ .*

**3.4. A family of nonconvergence examples.** In [KR] an example of a  $\psi$ -mixing system was built in which there exists a periodic point  $\omega$  such that  $\{S_{N_n^\omega}^{A_n^\omega}\}$  does not converge in distribution. This system was derived from a Bernoulli shift over the alphabet  $\{0, 1\}$  by using the group structure of  $\{0, 1\}$ . The next result extends this construction to every finite abelian group, producing a family of nonconvergence examples.

Let  $(G, +)$  be a finite abelian group, and let  $(p_g)_{g \in G} \subset (0, 1)$  be such that  $\sum_{g \in G} p_g = 1$ . Assume  $\Omega = G^{\mathbb{N}}$  and let  $\mathbb{P}$  be the measure on  $(\Omega, \mathcal{F})$  that satisfies

$$\mathbb{P}[g_0, \dots, g_{n-1}] = \prod_{j=0}^{n-1} p_{g_j}$$

for each  $g_0, \dots, g_{n-1} \in G$ . Then the coordinate projections from  $\Omega$  onto  $G$  are i.i.d. random elements.

Let  $N \geq 2$  be an integer, let  $\Phi : \Omega \rightarrow \Omega$  be such that

$$(\Phi\omega)_j = \omega_j + \dots + \omega_{j+N-1}$$

for each  $\omega \in \Omega$  and  $j \in \mathbb{N}$ , and set  $\mathbb{P}_0 = \mathbb{P} \circ \Phi^{-1}$ .

**Theorem 10.** (a)  $\mathbb{P}_0$  is  $T$ -invariant.

(b)  $\mathbb{P}_0$  is  $\psi$ -mixing.

(c) Assume there exist  $h \in G$  with  $p_h > p_g$  for all  $g \in G \setminus \{h\}$ . Then there exists  $s \in G$  such that for  $\omega \in \Omega$  with  $\omega_j = s$  for each  $j \geq 0$ , the limit  $\lim_n \mathbb{P}_0\{A_{n+1}^\omega \mid A_n^\omega\}$  does not exist.

From Theorem 10 we obtain:

**Corollary 11.** (a) In the conventional setup, it follows from Assertion (a) of Theorem 2 that  $\{S_{N_n^\omega}^{A_n^\omega}\}$  does not converge in distribution (where  $\omega$  is as in Assertion (c) of Theorem 10).

(b) It follows from Theorem 3 that the inverse Jacobian  $\frac{d\mathbb{P}_0}{d\mathbb{P}_0 \circ T}$  does not have a continuous version.

(c) It follows from Corollary 6 that  $\mathbb{P}_0$  is not a Gibbs measure corresponding to a Hölder continuous function.

Hence our construction provides a large class of  $\psi$ -mixing non Gibbs measures which seems to be new.

**3.5. Example of a pure Poisson limit distribution.** An example of a system will now be constructed in which there exists a periodic point  $\omega$  such that the limit distribution of  $\{S_{N_n}^{A_n^\omega}\}$  is pure Poisson.

Set  $\mathcal{A} = \mathbb{N}_+$  and  $\Omega = \mathcal{A}^\mathbb{N}$ , and let  $\mathbb{P}$  be the measure on  $(\Omega, \mathcal{F})$  such that

$$\mathbb{P}[a_0, \dots, a_{n-1}] = \prod_{j=0}^{n-1} 2^{-a_j}$$

for each  $a_0, \dots, a_{n-1} \in \mathcal{A}$ . Then the coordinate projections from  $\Omega$  onto  $\mathcal{A}$  are i.i.d. random variables.

Let  $\Omega_0 = \{0, 1\}^\mathbb{N}$ , let  $\mathcal{F}_0$  be the  $\sigma$ -algebra generated by cylinders, and let  $T_0 : \Omega_0 \rightarrow \Omega_0$  be the left-shift operator. For each  $a_1, a_2 \in \mathcal{A}$  set

$$\theta(a_1, a_2) = \begin{cases} 1 & , \text{ if } a_2 = a_1 + 1 \\ 0 & , \text{ otherwise} \end{cases}$$

Let  $\Theta : \Omega \rightarrow \Omega_0$  be such that  $(\Theta\omega)_j = \theta(\omega_j, \omega_{j+1})$  for each  $\omega \in \Omega$  and  $j \geq 0$ , and let  $\mathbb{P}_0 = \mathbb{P} \circ \Theta^{-1}$ .

**Theorem 12.** (a)  $\mathbb{P}_0$  is  $T_0$ -invariant.

(b)  $\mathbb{P}_0$  is  $\psi$ -mixing.

(c) Let  $\omega \in \Omega_0$  be such that  $\omega_j = 1$  for each  $j \geq 1$ , then  $\lim_n \mathbb{P}_0\{A_{n+1}^\omega \mid A_n^\omega\} = 0$ .

From Theorem 12 we obtain:

**Corollary 13.** (a) From Assertion (b) of Theorem 2 it follows that  $\mathcal{L}(S_{N_n}^{A_n^\omega}) \xrightarrow{d} \text{Pois}(t)$  as  $n \rightarrow \infty$ , where  $\omega$  is as in Assertion (c) of Theorem 12.

(b) From Corollary 6 it follows that  $\mathbb{P}_0$  is not a Gibbs measure corresponding to a Hölder continuous function.

#### 4. PROOF OF THEOREM 2

*Proof of Lemma 1:* It holds that

$$\begin{aligned} \lim_{n \rightarrow \infty} \rho_{A_n^\omega} &= \lim_{n \rightarrow \infty} \prod_{i=1}^{\ell} \mathbb{P}\{R^{(n+d_i\kappa(r))}/r \mid R^{n/r}\} = \\ &= \lim_{n \rightarrow \infty} \prod_{i=1}^{\ell} \prod_{j=1}^{d_i\kappa(r)/r} \frac{\mathbb{P}(R^{(n+jr)}/r)}{\mathbb{P}(R^{(n+(j-1)r)}/r)} = \lim_{n \rightarrow \infty} \prod_{i=1}^{\ell} \prod_{j=1}^{d_i\kappa(r)/r} \beta_{\omega, n+(j-1)r} = \beta_\omega^a \end{aligned}$$

and from (3.1) we see that  $\lim_{n \rightarrow \infty} \rho_{A_n^\omega} < 1$ .  $\square$

*Proof of Theorem 2:* For each  $n \geq r$  let  $W_n \sim \text{Pois}(t(1 - \rho_{A_n^\omega}))$ . From Theorem 2.3 in [KR] it follows that for each  $n \geq 1$  there exists a sequence of i.i.d. random

variables  $\eta_{n,1}, \eta_{n,2}, \dots$  independent of  $W_n$ , such that  $\mathbb{P}\{\eta_{n,1} \in \{1, \dots, [\frac{n}{r}]\}\} = 1$  and such that for  $Y_n = \sum_{j=1}^{W_n} \eta_{n,j}$

$$(4.1) \quad \lim_{n \rightarrow \infty} d_{TV}(\mathcal{L}(S_{N_n^\omega}^{A_n^\omega}), \mathcal{L}(Y_n)) = 0$$

(a) Assume that  $\lim_{n \rightarrow \infty} \rho_{A_n^\omega}$  does not exist, then there exists  $\epsilon > 0$  such that for each  $M \geq 1$ , there exist  $n > m > M$  with  $|\rho_{A_n^\omega} - \rho_{A_m^\omega}| > \epsilon$ . Let  $M \geq 1$  be such that  $d_{TV}(\mathcal{L}(S_{N_n^\omega}^{A_n^\omega}), \mathcal{L}(Y_n)) < \frac{e^{-t}t\epsilon}{3}$  for each  $n > M$ , then for each  $n > m > M$  with  $|\rho_{A_n^\omega} - \rho_{A_m^\omega}| > \epsilon$ ,

$$\begin{aligned} & |\mathbb{P}\{S_{N_n^\omega}^{A_n^\omega} = 0\} - \mathbb{P}\{S_{N_m^\omega}^{A_m^\omega} = 0\}| \geq \\ & \geq |\mathbb{P}\{Y_n = 0\} - \mathbb{P}\{Y_m = 0\}| - d_{TV}(\mathcal{L}(S_{N_n^\omega}^{A_n^\omega}), \mathcal{L}(Y_n)) - d_{TV}(\mathcal{L}(S_{N_m^\omega}^{A_m^\omega}), \mathcal{L}(Y_m)) > \\ & > |\exp(-t(1 - \rho_{A_n^\omega})) - \exp(-t(1 - \rho_{A_m^\omega}))| - \frac{e^{-t}t\epsilon}{3} - \frac{e^{-t}t\epsilon}{3} \geq \\ & \geq e^{-t}t|\rho_{A_n^\omega} - \rho_{A_m^\omega}| - 2\frac{e^{-t}t\epsilon}{3} > \frac{e^{-t}t\epsilon}{3} \end{aligned}$$

which shows that  $\{S_{N_n^\omega}^{A_n^\omega}\}_{n=1}^\infty$  does not converge in distribution.

(b) Assume that  $\beta_\omega = \lim_{n \rightarrow \infty} \beta_{\omega,n}$  exists, then from Lemma 1 it follows that the limit  $\rho_\omega = \lim_{n \rightarrow \infty} \rho_{A_n^\omega}$  exists, it is equal to  $\beta_\omega^a$ , and is strictly less than 1. Because of (4.1) it is enough to prove that  $\mathcal{L}(Y_n) \xrightarrow{d} PA(t(1 - \rho_\omega), \rho_\omega)$  as  $n \rightarrow \infty$ .

The main part of the proof will be showing that  $\mathcal{L}(\eta_{n,1}) \xrightarrow{d} Geo(\rho_\omega)$  as  $n \rightarrow \infty$ . To do this fix some integer  $b \geq 1$ . It is enough to show that  $\mathbb{P}\{\eta_{n,1} = b\} \xrightarrow{n \rightarrow \infty} Geo(\rho_\omega)\{b\}$ .

Since  $\rho_\omega < 1$  it follows that

$$1 - 2\beta_\omega^a + \beta_\omega^{2a} = 1 - 2\rho_\omega + \rho_\omega^2 > 0.$$

From this and from  $\psi_n \xrightarrow{n \rightarrow \infty} 0$  we see that there exist  $M \geq 1$  and  $\epsilon > 0$  such that

$$(4.2) \quad 1 - 2 \prod_{i=1}^{\ell} \prod_{j=1}^{d_i \kappa(r)/r} \beta_{\omega, n+(j-1)r} + \prod_{i=1}^{\ell} \prod_{j=1}^{2d_i \kappa(r)/r} \beta_{\omega, n+(j-1)r} - 4 \cdot 2^\ell \psi_n > \epsilon$$

for all  $n > M$ . By choosing  $M$  large enough and using (2.2) we may assume that  $N_n^\omega > 15d_\ell r n$ ,  $n > d_\ell \kappa(r)b$  and  $\psi_n < (3/2)^{1/(\ell+1)} - 1$  for each  $n \geq M$ .

Let  $n \geq M$ . We will use the notation from [KR] appearing in the statement and the proof of Theorem 2.3 there, with  $A = A_n^\omega$ . Namely, we set:  $R = [\omega_0, \dots, \omega_{r-1}]$ ,  $N = N_n^\omega$ ,  $K = 5d_\ell r n$ ,  $S_N = S_N^A$ ,  $X_k = X_k^A$  for each  $k \in \mathbb{N}_+$ ,  $\hat{X}_\alpha = 1_{\{K < \alpha \leq N\}} \cdot X_\alpha$

for each  $\alpha \in \mathbb{Z}$ ,  $n_0 = \lfloor \frac{n}{r} \rfloor$ ,  $\kappa = \kappa(r)$ ,  $\rho = \rho_A$ ,  $I_0 = \{K+1, \dots, N\} \times \{1, \dots, n_0\}$ ,

$$X_{\alpha,j} = (1 - \hat{X}_{\alpha-\kappa})(1 - \hat{X}_{\alpha+j\kappa}) \prod_{k=0}^{j-1} \hat{X}_{\alpha+k\kappa} \text{ for each } \alpha, j \in \mathbb{N}_+,$$

$\lambda_{\alpha,j} = E[X_{\alpha,j}]$  for each  $\alpha, j \in \mathbb{N}_+$ ,  $\lambda = \sum_{(\alpha,j) \in I_0} \lambda_{\alpha,j}$ ,  $\lambda_j = \lambda^{-1} \sum_{\alpha=K+1}^N \lambda_{\alpha,j}$  for each  $1 \leq j \leq n_0$ , and  $s = t(1 - \rho)$ . We recall that in the proof of Theorem 2.3 in [KR], the i.i.d. random variables  $\eta_{n,1}, \eta_{n,2}, \dots$  were chosen so that  $\mathbb{P}\{\eta_{n,1} = j\} = \lambda_j$  for each  $1 \leq j \leq n_0$ . First we shall need to bound  $\lambda$  from below.

*Claim:* It holds that  $\lambda > \frac{t\epsilon}{2}$ .

*Proof:* Let  $K < \alpha \leq N$ , then from Lemma 3.2 in [KR] it follows that

$$\begin{aligned} \lambda_{\alpha,1} &= E[X_{\alpha,1}] \geq E[(1 - X_{\alpha-\kappa})(1 - X_{\alpha+\kappa})X_{\alpha}] = \\ &= E[X_{\alpha}] - E[X_{\alpha-\kappa}X_{\alpha}] - E[X_{\alpha+\kappa}X_{\alpha}] + E[X_{\alpha-\kappa}X_{\alpha+\kappa}X_{\alpha}] = \\ &= \mathbb{P}\left(\bigcap_{i=1}^{\ell} T^{-d_i\alpha}(A)\right) - \mathbb{P}\left(\bigcap_{i=1}^{\ell} T^{-d_i(\alpha-\kappa)}(R^{(n+d_i\kappa)/r})\right) - \\ &\quad - \mathbb{P}\left(\bigcap_{i=1}^{\ell} T^{-d_i\alpha}(R^{(n+d_i\kappa)/r})\right) + \mathbb{P}\left(\bigcap_{i=1}^{\ell} T^{-d_i(\alpha-\kappa)}(R^{(n+2d_i\kappa)/r})\right) \geq \\ &\geq (\mathbb{P}(A))^{\ell}(1-2^{\ell}\psi_n) - 2\left(\prod_{i=1}^{\ell} \mathbb{P}(R^{(n+d_i\kappa)/r})\right)(1+2^{\ell}\psi_n) + (1-2^{\ell}\psi_n)\left(\prod_{i=1}^{\ell} \mathbb{P}(R^{(n+2d_i\kappa)/r})\right) \geq \\ &\geq (\mathbb{P}(A))^{\ell} - 2\prod_{i=1}^{\ell} \mathbb{P}(R^{(n+d_i\kappa)/r}) + \prod_{i=1}^{\ell} \mathbb{P}(R^{(n+2d_i\kappa)/r}) - 4 \cdot 2^{\ell}\psi_n(\mathbb{P}(A))^{\ell}. \end{aligned}$$

Hence, from (4.2),

$$\begin{aligned} (\mathbb{P}(A))^{-\ell} \lambda_{\alpha,1} &\geq 1 - 2\prod_{i=1}^{\ell} \frac{\mathbb{P}(R^{(n+d_i\kappa)/r})}{\mathbb{P}(A)} + \prod_{i=1}^{\ell} \frac{\mathbb{P}(R^{(n+2d_i\kappa)/r})}{\mathbb{P}(A)} - 4 \cdot 2^{\ell}\psi_n = \\ &= 1 - 2\prod_{i=1}^{\ell} \prod_{j=1}^{d_i\kappa/r} \frac{\mathbb{P}(R^{(n+jr)/r})}{\mathbb{P}(R^{(n+(j-1)r)/r})} + \prod_{i=1}^{\ell} \prod_{j=1}^{2d_i\kappa/r} \frac{\mathbb{P}(R^{(n+jr)/r})}{\mathbb{P}(R^{(n+(j-1)r)/r})} - 4 \cdot 2^{\ell}\psi_n = \\ &= 1 - 2\prod_{i=1}^{\ell} \prod_{j=1}^{d_i\kappa/r} \beta_{\omega, n+(j-1)r} + \prod_{i=1}^{\ell} \prod_{j=1}^{2d_i\kappa/r} \beta_{\omega, n+(j-1)r} - 4 \cdot 2^{\ell}\psi_n > \epsilon, \end{aligned}$$

and so

$$\lambda \geq \sum_{\alpha=K+1}^N \lambda_{\alpha,1} \geq (N-K)(\mathbb{P}(A))^{\ell}\epsilon > \frac{t\epsilon}{2}.$$

We resume the proof of Assertion (b) of Theorem 2. From  $\lambda^{-1} \leq \frac{2}{t\epsilon}$  it follows that

$$\begin{aligned} & |\mathbb{P}\{\eta_{n,1} = b\} - Geo(\rho_\omega)\{b\}| = |\lambda_b - (1 - \rho_\omega)\rho_\omega^{b-1}| = \\ & = \lambda^{-1} \left| \sum_{\alpha=K+1}^N \lambda_{\alpha,b} - \lambda(1 - \rho_\omega)\rho_\omega^{b-1} \right| \leq \frac{2}{t\epsilon} \left| \sum_{\alpha=K+1}^N \lambda_{\alpha,b} - s(1 - \rho_\omega)\rho_\omega^{b-1} \right| + \frac{2}{t\epsilon} |\lambda - s|. \end{aligned}$$

From the inequality (5.5) in the proof of Theorem 2.3 in [KR], it follows that  $|\lambda - s|$  tends to 0 as  $n \rightarrow \infty$ . Hence in order to prove that  $\mathcal{L}(\eta_{n,1}) \xrightarrow{d} Geo(\rho_\omega)$ , it is enough to show that

$$\lim_{n \rightarrow \infty} \left| \sum_{\alpha=K+1}^N \lambda_{\alpha,b} - s(1 - \rho_\omega)\rho_\omega^{b-1} \right| = 0.$$

Also, since

$$s = t(1 - \rho_{A_n^\omega}) \xrightarrow{n \rightarrow \infty} t(1 - \rho_\omega)$$

it is enough to show that

$$\lim_{n \rightarrow \infty} \left| \sum_{\alpha=K+1}^N \lambda_{\alpha,b} - t(1 - \rho_\omega)^2 \rho_\omega^{b-1} \right| = 0.$$

We also have

$$\begin{aligned} & \left| \sum_{\alpha=K+1}^N \lambda_{\alpha,b} - t(1 - \rho_\omega)^2 \rho_\omega^{b-1} \right| \leq \\ & \leq (\mathbb{P}(A))^\ell + \left| \sum_{\alpha=K+1}^N \lambda_{\alpha,b} - N(\mathbb{P}(A))^\ell (1 - \rho_\omega)^2 \rho_\omega^{b-1} \right| \leq \\ & \leq 5K\mathbb{P}(A) + \sum_{\alpha=K+\kappa+1}^{N-b\kappa} |\lambda_{\alpha,b} - (\mathbb{P}(A))^\ell (1 - \rho_\omega)^2 \rho_\omega^{b-1}|, \end{aligned}$$

and from (2.2),

$$K\mathbb{P}(A) \leq 5d_\ell r n \cdot e^{-\Gamma n} \xrightarrow{n \rightarrow \infty} 0.$$

Hence, it is enough to show that

$$\lim_{n \rightarrow \infty} \sum_{\alpha=K+\kappa+1}^{N-b\kappa} |\lambda_{\alpha,b} - (\mathbb{P}(A))^\ell (1 - \rho_\omega)^2 \rho_\omega^{b-1}| = 0.$$

This will be established once we prove the following:

*Claim:* It holds that

$$(4.3) \quad \lim_{n \rightarrow \infty} (\mathbb{P}(A))^{-\ell} \sup\{|\lambda_{\alpha,b} - (\mathbb{P}(A))^\ell (1 - \rho_\omega)^2 \rho_\omega^{b-1}| : K + \kappa < \alpha \leq N - b\kappa\} = 0.$$

*Proof:* Let  $K + \kappa < \alpha \leq N - b\kappa$ , then

$$\begin{aligned}
(4.4) \quad & |\lambda_{\alpha,b} - (\mathbb{P}(A))^\ell (1 - \rho_\omega)^2 \rho_\omega^{b-1}| = \\
& = |E[(1 - X_{\alpha-\kappa})(1 - X_{\alpha+b\kappa}) \prod_{k=0}^{b-1} X_{\alpha+k\kappa}] - (\mathbb{P}(A))^\ell (1 - \rho_\omega)^2 \rho_\omega^{b-1}| = \\
& \leq |E[\prod_{k=0}^{b-1} X_{\alpha+k\kappa}] - (\mathbb{P}(A))^\ell \rho_\omega^{b-1}| + |E[\prod_{k=-1}^{b-1} X_{\alpha+k\kappa}] - (\mathbb{P}(A))^\ell \rho_\omega^b| + \\
& + |E[\prod_{k=0}^b X_{\alpha+k\kappa}] - (\mathbb{P}(A))^\ell \rho_\omega^b| + |E[\prod_{k=-1}^b X_{\alpha+k\kappa}] - (\mathbb{P}(A))^\ell \rho_\omega^{b+1}| = \\
& = \delta(\alpha, 0, b-1) + \delta(\alpha, -1, b-1) + \delta(\alpha, 0, b) + \delta(\alpha, -1, b)
\end{aligned}$$

where

$$\delta(\alpha, q, p) = |E[\prod_{k=q}^p X_{\alpha+k\kappa}] - (\mathbb{P}(A))^\ell \rho_\omega^{p-q}|$$

for each  $K + \kappa < \alpha \leq N - b\kappa$ ,  $q \in \{-1, 0\}$  and  $p \in \{b-1, b\}$ . Fix such  $\alpha$ ,  $q$  and  $p$ , then

$$\begin{aligned}
(4.5) \quad & \delta(\alpha, q, p) \leq |E[\prod_{k=q}^p X_{\alpha+k\kappa}] - \prod_{i=1}^\ell \mathbb{P}(R^{(n+(p-q)d_i\kappa)/r})| + \\
& + |\prod_{i=1}^\ell \mathbb{P}(R^{(n+(p-q)d_i\kappa)/r}) - (\mathbb{P}(A))^\ell \prod_{i=1}^\ell \prod_{k=1}^{p-q} \mathbb{P}\{R^{(n+kd_i\kappa)/r} \mid R^{(n+(k-1)d_i\kappa)/r}\}| + \\
& + |(\mathbb{P}(A))^\ell \prod_{i=1}^\ell \prod_{k=1}^{p-q} \mathbb{P}\{R^{(n+kd_i\kappa)/r} \mid R^{(n+(k-1)d_i\kappa)/r}\} - (\mathbb{P}(A))^\ell \rho_\omega^{p-q}| = \Lambda_1 + \Lambda_2 + \Lambda_3
\end{aligned}$$

where  $\Lambda_1$ ,  $\Lambda_2$  and  $\Lambda_3$  denote the first, second and third terms respectively, and if  $p = q$  then the multiplication from 1 to  $p - q$  equals 1 by definition. From Lemma 3.2 in [KR] it follows that

$$\begin{aligned}
\Lambda_1 & = |\mathbb{P}(\bigcap_{i=1}^\ell T^{-(\alpha+q\kappa)d_i}(R^{(n+(p-q)d_i\kappa)/r})) - \prod_{i=1}^\ell \mathbb{P}(R^{(n+(p-q)d_i\kappa)/r})| \leq \\
& \leq ((1 + \psi_n)^\ell - 1) \prod_{i=1}^\ell \mathbb{P}(R^{(n+(p-q)d_i\kappa)/r}) \leq 2^\ell \psi_n (\mathbb{P}(A))^\ell.
\end{aligned}$$



The term  $\Lambda_2$  vanishes and

$$\begin{aligned}\Lambda_3 &= (\mathbb{P}(A))^\ell \left| \prod_{i=1}^{\ell} \prod_{k=1}^{p-q} \prod_{j=1+(k-1)d_i\kappa/r}^{kd_i\kappa/r} \frac{\mathbb{P}(R^{(n+jr)/r})}{\mathbb{P}(R^{(n+(j-1)r)/r})} - \rho_\omega^{p-q} \right| = \\ &= (\mathbb{P}(A))^\ell \left| \prod_{i=1}^{\ell} \prod_{k=1}^{p-q} \prod_{j=1+(k-1)d_i\kappa/r}^{kd_i\kappa/r} \beta_{\omega, n+(j-1)r} - \rho_\omega^{p-q} \right|.\end{aligned}$$

Hence from (4.4) and (4.5) it follows that

$$\begin{aligned}(4.6) \quad & (\mathbb{P}(A))^{-\ell} \sup\{|\lambda_{\alpha,b} - (\mathbb{P}(A))^\ell (1 - \rho_\omega)^2 \rho_\omega^{b-1}| : K + \kappa < \alpha \leq N - b\kappa\} \leq \\ & \leq 4(\mathbb{P}(A))^{-\ell} \sup\{\delta(\alpha, q, p) : K + \kappa < \alpha \leq N - b\kappa, q \in \{-1, 0\}, p \in \{b-1, b\}\} \leq \\ & \leq 4 \cdot 2^\ell \psi_n + 4 \left| \prod_{i=1}^{\ell} \prod_{k=1}^{p-q} \prod_{j=1+(k-1)d_i\kappa/r}^{kd_i\kappa/r} \beta_{\omega, n+(j-1)r} - \rho_\omega^{p-q} \right|.\end{aligned}$$

From  $\beta_\omega = \lim_{n \rightarrow \infty} \beta_{\omega, n}$  it follows that

$$\begin{aligned}\lim_{n \rightarrow \infty} \prod_{i=1}^{\ell} \prod_{k=1}^{p-q} \prod_{j=1+(k-1)d_i\kappa/r}^{kd_i\kappa/r} \beta_{\omega, n+(j-1)r} &= \\ &= \prod_{i=1}^{\ell} \prod_{k=1}^{p-q} \prod_{j=1+(k-1)d_i\kappa/r}^{kd_i\kappa/r} \beta_\omega = \prod_{k=1}^{p-q} \prod_{i=1}^{\ell} \beta_\omega^{(d_i\kappa)/r} = \rho_\omega^{p-q}\end{aligned}$$

so from  $\psi_n \xrightarrow{n \rightarrow \infty} 0$  and from (4.6) we obtain (4.3).

We have thus proven that  $\mathcal{L}(\eta_{n,1}) \xrightarrow{d} Geo(\rho_\omega)$  as  $n \rightarrow \infty$ , and we can now finally show that  $\mathcal{L}(Y_n) \xrightarrow{d} PA(t(1 - \rho_\omega), \rho_\omega)$  as  $n \rightarrow \infty$ . Let  $x \in \mathbb{R}$ , then from Theorem 26.3 in [B1] it follows that

$$\lim_{n \rightarrow \infty} \varphi_{\eta_{n,1}}(x) = \varphi_{Geo(\rho_\omega)}(x),$$

where recall that we denote by  $\varphi_\xi$  the characteristic function of a random variable or measure  $\xi$ . Hence, by (2.3) and (2.4),

$$\begin{aligned}\lim_{n \rightarrow \infty} \varphi_{\mathcal{L}(Y_n)}(x) &= \lim_{n \rightarrow \infty} \varphi_{CP(t(1-\rho_{A_n^\omega}), \mathcal{L}(\eta_{n,1}))}(x) = \\ &= \lim_{n \rightarrow \infty} \exp(t(1 - \rho_{A_n^\omega})(\varphi_{\eta_{n,1}}(x) - 1)) = \\ &= \exp(t(1 - \rho_\omega)(\varphi_{Geo(\rho_\omega)}(x) - 1)) = \varphi_{PA(t(1-\rho_\omega), \rho_\omega)}(x).\end{aligned}$$

Now another application of Theorem 26.3 gives  $\mathcal{L}(Y_n) \xrightarrow{d} PA(t(1 - \rho_\omega), \rho_\omega)$  as  $n \rightarrow \infty$ , which completes the proof of the theorem.  $\square$

## 5. PROOF OF THE RESULTS OF SECTION 3.2

In this section we will write  $T$  for the restriction of  $T$  to  $\Omega_S$  and for each  $w \in \mathcal{A}_S^*$  we will write  $[w]$  for  $[w] \cap \Omega_S$ .

*Proof of Theorem 3:* Set  $w = \omega_0 \cdot \dots \cdot \omega_{r-1}$  and for each  $n \geq 1$  and  $0 \leq j < r$  set

$$E_{n,j} = [\omega_{r-1-j} \cdot \dots \cdot \omega_{r-1} \cdot w^{n/r}]$$

then

$$T(E_{n,j}) = \begin{cases} [w^{n/r}] & , \text{ if } j = 0 \\ [\omega_{r-j} \cdot \dots \cdot \omega_{r-1} \cdot w^{n/r}] & , \text{ if } 0 < j < r \end{cases}.$$

Hence, given  $n \geq 1$ ,

$$(5.1) \quad \beta_{\omega,n} = \frac{\mathbb{P}([w^{(n+r)/r}])}{\mathbb{P}([w^{n/r}])} = \prod_{j=0}^{r-1} \frac{\mathbb{P}(E_{n,j})}{\mathbb{P}(T(E_{n,j}))} = \prod_{j=0}^{r-1} \frac{1}{\mathbb{P} \circ T(E_{n,j})} \int_{E_{n,j}} J d\mathbb{P} \circ T.$$

Let  $0 \leq j < r$  and  $\epsilon > 0$ . Since  $J$  is continuous, since  $T^{r-1-j}(\omega) \in E_{n,j}$  for all  $n \geq 1$ , and since

$$\sup\{d(\gamma, \eta) : \gamma, \eta \in E_{n,j}\} \xrightarrow{n \rightarrow \infty} 0$$

(where  $d$  is the metric defined in section 2.1), it follows that there exist  $N \geq 1$  with

$$|J(T^{r-1-j}(\omega)) - J(\gamma)| < \epsilon$$

for each  $n \geq N$  and  $\gamma \in E_{n,j}$ . From this it follows that for each  $n \geq N$

$$\begin{aligned} \left| \frac{1}{\mathbb{P} \circ T(E_{n,j})} \int_{E_{n,j}} J d\mathbb{P} \circ T - J(T^{r-1-j}(\omega)) \right| &\leq \\ &\leq \frac{1}{\mathbb{P} \circ T(E_{n,j})} \int_{E_{n,j}} |J(\gamma) - J(T^{r-1-j}(\omega))| d\mathbb{P} \circ T(\gamma) \leq \epsilon, \end{aligned}$$

and so

$$(5.2) \quad \lim_n \frac{1}{\mathbb{P} \circ T(E_{n,j})} \int_{E_{n,j}} J d\mathbb{P} \circ T = J(T^{r-1-j}(\omega)).$$

The theorem now follows from (5.1) and (5.2).  $\square$

*Proof of Theorem 5:* Let  $C(\Omega_S)$  be the Banach space of all continuous functions from  $\Omega_S$  to  $\mathbb{R}$ . Let  $L_\phi : C(\Omega_S) \rightarrow C(\Omega_S)$  be the Ruelle (transfer) operator associated with  $\phi$ , i.e.

$$(L_\phi f)(x) = \sum_{y \in T^{-1}\{x\}} e^{\phi(y)} f(y)$$

for each  $f \in C(\Omega_S)$  and  $x \in \Omega_S$ . From the construction of the Gibbs measure  $\mathbb{P}_\phi$  carried out in sections B and C of Chapter 1 in [B2], it follows that there exist  $h \in C(\Omega_S)$  with  $h > 0$  and a probability measure  $\nu$  on  $(\Omega_S, \mathcal{F}_S)$  such that  $L_\phi h = e^P h$ ,  $L_\phi^* \nu = e^P \nu$  and  $d\mathbb{P}_\phi = h d\nu$ . Here  $L_\phi^*$  is the adjoint operator of  $L_\phi$ ,

which satisfies

$$\int f d(L_\phi^* \mu) = \int L_\phi f d\mu$$

for each finite measure  $\mu$  on  $(\Omega_S, \mathcal{F}_S)$ , and  $f \in C(\Omega_S)$ .

Let  $\theta$  be the measure on  $(\Omega_S, \mathcal{F}_S)$  with

$$\theta(E) = \int_E e^{P-\phi} \cdot \frac{h \circ T}{h} d\mathbb{P}$$

for each  $E \in \mathcal{F}_S$ . Let  $n \geq 1$  and  $a_0 \cdot \dots \cdot a_{n-1} = w \in \mathcal{A}_S^*$ , then

$$\begin{aligned} (5.3) \quad \theta[w] &= \int_{[w]} e^{P-\phi} \cdot h \circ T d\nu = \int_{[w]} e^{-\phi} \cdot h \circ T d(L_\phi^* \nu) = \\ &= \int_{[w]} L_\phi(1_{[w]} \cdot e^{-\phi} \cdot h \circ T) d\nu = \int \sum_{y \in T^{-1}x} e^{\phi(y)} 1_{[w]}(y) e^{-\phi(y)} h(Ty) d\nu(x) = \\ &= \int 1_{[w]}(a_0 \cdot x) h(x) d\nu(x) = \int 1_{[a_1 \cdot \dots \cdot a_{n-1}]}(x) h(x) d\nu(x) = \mathbb{P}[a_1 \cdot \dots \cdot a_{n-1}] = \mathbb{P} \circ T[w]. \end{aligned}$$

Set  $\mathcal{L} = \{E \in \mathcal{F}_S : \mathbb{P} \circ T(E) = \theta(E)\}$  and  $\mathcal{P} = \{[w] : w \in \mathcal{A}_S^*\} \cup \{\emptyset\}$ . From (5.3) it follows that  $\Omega_S \in \mathcal{L}$ , and so it is easy to check that  $\mathcal{L}$  is a  $\lambda$ -system. Also,  $\mathcal{P}$  is a  $\pi$ -system and from (5.3) we get that  $\mathcal{P} \subset \mathcal{L}$ . From the  $\pi - \lambda$  theorem (see [B1]) it follows that  $\mathcal{F}_S = \sigma(\mathcal{P}) \subset \mathcal{L}$ , which shows that  $\mathbb{P} \circ T = \theta$ . This shows that  $d\mathbb{P} = e^{\phi-P} \cdot \frac{h}{h \circ T} d\mathbb{P} \circ T$ , as required.  $\square$

*Proof of Corollary 6:* From Theorem 5 and since  $T^r \omega = \omega$ , it follows that

$$\prod_{j=0}^{r-1} J(T^j \omega) = \prod_{j=0}^{r-1} e^{\phi(T^j \omega) - P} \frac{h(T^j \omega)}{h(T^{j+1} \omega)} = \exp\left(\sum_{j=0}^{r-1} (\phi(T^j \omega) - P)\right)$$

Now since  $J = e^{\phi-P} \cdot \frac{h}{h \circ T}$  is a continuous function, the claim follows from Corollary 4.  $\square$

We now turn to the proof of Theorem 7. We shall need some additional notations. For each  $a \in \mathcal{A}$  set  $K_a = f_a(I_a)$  and let  $g_a : K_a \rightarrow I_a$  be the inverse function of  $f_a$ . From the assumptions on  $f$  it follows that  $g_a \in C^1(K_a)$ , and  $g_a$  is strictly monotonic. Let  $a_0 \cdot \dots \cdot a_{r-1} = w \in \mathcal{A}_S^*$ , then

$$g_{a_j}(K_{a_j}) = I_{a_j} \subset f_{a_{j-1}}(I_{a_{j-1}}) = K_{a_{j-1}}$$

for each  $1 \leq j < r$ , so the function  $g_w : K_{a_{r-1}} \rightarrow I$  given by  $g_w = g_{a_0} \circ \dots \circ g_{a_{r-1}}$  is well defined. For such a  $w$  we set  $I_w = g_w(K_{a_{r-1}})$ .

The following lemmas will be needed in the proof of Theorem 7.

**Lemma.**  $\Theta(\omega)$  equals the unique element in  $\bigcap_{n \geq 1} I_{\omega_0 \cdot \dots \cdot \omega_{n-1}}$  for each  $\omega \in \Omega_S$ , and

$$(5.4) \quad \Theta(a \cdot \omega) = g_a(\Theta(\omega)) \text{ for each } a \in \mathcal{A} \text{ and } \omega \in T[a].$$

*Proof:* First we shall prove by induction on  $n$  that

$$(5.5) \quad \bigcap_{j=0}^{n-1} f^{-j}(U_{a_j}) = g_{a_0 \dots a_{n-2}}(U_{a_{n-1}})$$

for each  $n \geq 1$  and  $a_0 \dots a_{n-1} \in \mathcal{A}_S^*$ . This is clear for  $n = 1$  since both sides are equal to  $U_{a_0}$ . Assume (5.5) holds for  $n \geq 1$  and let  $a_0 \dots a_n \in \mathcal{A}_S^*$ , then

$$\begin{aligned} \bigcap_{j=0}^n f^{-j}(U_{a_j}) &= U_{a_0} \cap f^{-1}\left(\bigcap_{j=0}^{n-1} f^{-j}(U_{a_{j+1}})\right) = \\ &= U_{a_0} \cap f^{-1}(g_{a_1 \dots a_{n-1}}(U_{a_n})) = f_a^{-1}(g_{a_1 \dots a_{n-1}}(U_{a_n})) = g_{a_0 \dots a_{n-1}}(U_{a_n}) \end{aligned}$$

so (5.5) holds for all  $n \geq 1$ .

From this it follows that for each  $\omega \in \Omega_S$

$$\bigcap_{n \geq 1} \overline{\bigcap_{j=0}^{n-1} f^{-j}(U_{\omega_j})} = \bigcap_{n \geq 1} \overline{g_{\omega_0 \dots \omega_{n-2}}(U_{\omega_{n-1}})} = \bigcap_{n \geq 1} I_{\omega_0 \dots \omega_{n-1}}$$

and so, from the definition of  $\Theta$  found in Proposition 1.2 in [S],  $\Theta(\omega)$  equals the unique element in  $\bigcap_{n \geq 1} I_{\omega_0 \dots \omega_{n-1}}$ . Let  $a \in \mathcal{A}$ ,  $\omega \in T[a]$  and  $\gamma = a \cdot \omega$ , then since

$$\bigcap_{n \geq 1} I_{\gamma_0 \dots \gamma_{n-1}} = g_a\left(\bigcap_{n \geq 1} I_{\omega_0 \dots \omega_{n-1}}\right)$$

it follows that  $\Theta(\gamma) = g_a(\Theta(\omega))$ , and the lemma is proved.  $\square$

**Lemma.** Let  $a_0 \dots a_{n-1} = w \in \mathcal{A}_S^*$ , then

$$(5.6) \quad \Phi^{-1}[w] = \tilde{I} \cap I_w$$

*Proof:* For each  $0 \leq m \leq n$  set

$$F_m = \{x \in \tilde{I} : f^j(x) \in I_{a_j} \text{ for each } m \leq j < n\}$$

and

$$E_m = \{x \in \tilde{I} : f_{a_{j-1}} \circ \dots \circ f_{a_0}(x) \in I_{a_j} \text{ for each } 0 \leq j < m\}.$$

Clearly  $F_m \cap E_m = F_{m+1} \cap E_{m+1}$  for each  $0 \leq m < n$ . Also  $\Phi^{-1}[w] = F_0 \cap E_0$ , since

$$\begin{aligned} \Phi^{-1}[w] &= \{x \in \tilde{I} : \xi_j(x) = a_j \text{ for all } 0 \leq j < n\} = \\ &= \{x \in \tilde{I} : f^j(x) \in U_{a_j} \text{ for all } 0 \leq j < n\} = \\ &= \{x \in \tilde{I} : f^j(x) \in I_{a_j} \text{ for all } 0 \leq j < n\} \end{aligned}$$

which shows that  $\Phi^{-1}[w] = F_n \cap E_n$ , and so

$$\begin{aligned}\Phi^{-1}[w] &= \{x \in \tilde{I} : f_{a_{j-1}} \circ \dots \circ f_{a_0}(x) \in I_{a_j} \text{ for each } 0 \leq j < n\} = \\ &= \{x \in \tilde{I} : x \in g_{a_0} \circ \dots \circ g_{a_j}(K_{a_j}) \text{ for all } 0 \leq j < n\}.\end{aligned}$$

For each  $1 \leq j < n$  it holds that  $g_{a_j}(K_{a_j}) \subset K_{a_{j-1}}$ , so

$$g_{a_0} \circ \dots \circ g_{a_j}(K_{a_j}) \subset g_{a_0} \circ \dots \circ g_{a_{j-1}}(K_{a_{j-1}})$$

and so

$$\Phi^{-1}[w] = \{x \in \tilde{I} : x \in g_{a_0} \circ \dots \circ g_{a_{n-1}}(K_{a_{n-1}})\} = \tilde{I} \cap I_w$$

as desired.  $\square$

**Lemma.** *For every  $\mathbb{P} \circ T$ -integrable function  $h$  it holds that*

$$(5.7) \quad \int h \, d\mathbb{P} \circ T = \sum_{a \in \mathcal{A}} \int_{T[a]} h(a \cdot \omega) \, d\mathbb{P}(\omega).$$

*Proof:* Let  $E \in \mathcal{F}_S$ , then

$$\begin{aligned}\int 1_E \, d\mathbb{P} \circ T &= \sum_{a \in \mathcal{A}} \mathbb{P}(T(E \cap [a])) = \\ &= \sum_{a \in \mathcal{A}} \int 1_{T(E \cap [a])}(\omega) \, d\mathbb{P}(\omega) = \sum_{a \in \mathcal{A}} \int_{T[a]} 1_E(a \cdot \omega) \, d\mathbb{P}(\omega)\end{aligned}$$

hence the lemma holds for indicator functions. From linearity the lemma follows for simple functions, from the monotone convergence theorem it follows for positive functions, and by writing  $h = (h \vee 0) - (-h \vee 0)$  the lemma follows for every  $\mathbb{P} \circ T$ -integrable function  $h$ .  $\square$

*Proof of Theorem 7:* For each  $\omega \in \Omega_S$  set  $h(\omega) = \frac{p(\Theta(\omega))}{f'_{\omega_0}(\Theta(\omega))p(\Theta(T\omega))}$ . Let  $X = \Phi(\tilde{I})$ , then  $\Theta : X \rightarrow \tilde{I}$  is a bijection whose inverse is  $\Phi$ , and also

$$\mathbb{P}(X) = \mu(\Phi^{-1}(\Phi(\tilde{I}))) = \mu(\tilde{I}) = 1.$$

Let  $a_0 \cdot \dots \cdot a_{r-1} = w \in \mathcal{A}_S^*$ , and set  $\alpha = \int_{[w]} h(\omega) \, d\mathbb{P} \circ T$ . From (5.7),

$$\begin{aligned}(5.8) \quad \alpha &= \sum_{a \in \mathcal{A}} \int_{T[a]} 1_{[w]}(a \cdot \omega) h(a \cdot \omega) \, d\mathbb{P}(\omega) = \\ &= \int_{T[a_0]} 1_{[w]}(a_0 \cdot \omega) h(a_0 \cdot \omega) \, d\mathbb{P}(\omega).\end{aligned}$$

It holds that

$$\Theta(T[a_0] \cap X) = \Theta\{\omega \in X : S_{a_0, \omega_0} = 1\} = \tilde{I} \cap (\bigcup \{U_b : S_{a_0, b} = 1\}) = \tilde{I} \cap K_{a_0}$$

so  $\omega \in T[a_0] \cap X$  if and only if  $\Theta(\omega) \in \tilde{I} \cap K_{a_0}$ . From (5.6) it follows that

$$\Theta([w] \cap X) = \Phi^{-1}[w] = I_w \cap \tilde{I},$$

so  $a_0 \cdot \omega \in [w] \cap X$  if and only if  $\Theta(a_0 \cdot \omega) \in I_w \cap \tilde{I}$  for each  $\omega \in T[a_0]$ . Also, if  $\omega \in T[a_0] \cap X$  then  $a_0 \cdot \omega \in X$ . From this, from  $\mathbb{P}(X) = 1$ , and from (5.8),

$$\alpha = \int_X 1_{K_{a_0}}(\Theta(\omega)) \cdot 1_{I_w}(\Theta(a_0 \cdot \omega)) \cdot \frac{p(\Theta(a_0 \cdot \omega))}{f'_{a_0}(\Theta(a_0 \cdot \omega))p(\Theta(\omega))} d\mathbb{P}(\omega).$$

This together with (5.4) gives

$$\begin{aligned} \alpha &= \int_X 1_{K_{a_0}}(\Theta(\omega)) \cdot 1_{I_w}(g_{a_0}(\Theta(\omega))) \cdot \frac{p(g_{a_0}(\Theta(\omega)))}{f'_{a_0}(g_{a_0}(\Theta(\omega)))p(\Theta(\omega))} d\mathbb{P}(\omega) = \\ &= \int_{\tilde{I}} 1_{K_{a_0}}(x) \cdot 1_{I_w}(g_{a_0}(x)) \cdot \frac{p(g_{a_0}(x))}{f'_{a_0}(g_{a_0}(x))p(x)} d\mathbb{P} \circ \Theta^{-1}(x) \end{aligned}$$

and since  $g_{a_0}^{-1}(I_w) \subset K_{a_0}$ ,

$$\alpha = \int_{\tilde{I}} 1_{g_{a_0}^{-1}(I_w)}(x) \cdot \frac{p(g_{a_0}(x))}{f'_{a_0}(g_{a_0}(x))p(x)} d\mathbb{P} \circ \Theta^{-1}(x).$$

Since  $\mathbb{P} = \mu \circ \Phi^{-1}$  and  $d\mu = p dm$ , this shows that

$$\begin{aligned} \alpha &= \int_{\tilde{I}} 1_{g_{a_0}^{-1}(I_w)}(x) \cdot \frac{p(g_{a_0}(x))}{f'_{a_0}(g_{a_0}(x))p(x)} d\mu(x) = \\ &= \int 1_{g_{a_0}^{-1}(I_w)}(x) \cdot \frac{p(g_{a_0}(x))}{f'_{a_0}(g_{a_0}(x))} dm(x), \end{aligned}$$

and since  $g'_{a_0} = \frac{1}{f'_{a_0} \circ g_{a_0}}$ ,

$$\begin{aligned} \alpha &= \int 1_{g_{a_0}^{-1}(I_w)}(x) \cdot p(g_{a_0}(x))g'_{a_0}(x) dm(x) = \\ &= \int_{I_w} p(y) dm(y) = \mu([I_w] \cap \tilde{I}) = \mu(\Phi^{-1}[w]) = \mathbb{P}[w]. \end{aligned}$$

This holds for any  $w \in \mathcal{A}_S^*$ , hence from the  $\pi - \lambda$  theorem (see the end of the proof of Theorem 5) it follows that  $\frac{d\mathbb{P}}{d\mathbb{P} \circ T} = h$ , as required.  $\square$

*Proof of Corollary 8:* From Theorem 7 and since  $T^r \omega = \omega$ , it follows that

$$\prod_{j=0}^{r-1} J(T^j \omega) = \prod_{j=0}^{r-1} \frac{p(\Theta(T^j \omega))}{f'_{\omega_j}(\Theta(T^j \omega))p(\Theta(T^{j+1} \omega))} = \left( \prod_{j=0}^{r-1} f'_{\omega_j}(\Theta(T^j \omega)) \right)^{-1}.$$

Now since the map  $\omega \rightarrow \frac{p(\Theta(\omega))}{f'_{\omega_0}(\Theta(\omega))p(\Theta(T\omega))}$  is continuous, the claim follows from Corollary 4.  $\square$

## 6. PROOF OF THEOREM 9

*Proof of Assertion (a):* Let  $\epsilon > 0$ , let  $\Gamma > 0$  be as in (2.2), and let  $M \geq 1$  be such that  $ne^{-\Gamma n} < 1$  for all  $n \geq M$ . Let  $n \geq M$ , then from Lemma 3.2 in [KR] it follows that for each  $n \leq k \leq N_n^\omega$ ,

$$\mathbb{P}\{X_k^{A_n^\omega} = 1\} \leq (1 + \psi_0)^\ell (\mathbb{P}(A_n^\omega))^\ell.$$

Also, for each  $1 \leq k < n$  it follows from (2.2) that

$$\mathbb{P}\{X_k^{A_n^\omega} = 1\} \leq \mathbb{P}(A_n^\omega) \leq e^{-\Gamma n}.$$

Hence,

$$(6.1) \quad E[S_{N_n^\omega}^{A_n^\omega}] = \sum_{k=1}^{N_n^\omega} \mathbb{P}\{X_k^{A_n^\omega} = 1\} \leq ne^{-\Gamma n} + N_n^\omega (1 + \psi_0)^\ell (\mathbb{P}(A_n^\omega))^\ell \leq 1 + (1 + \psi_0)^\ell t.$$

Let  $b \geq \frac{1+(1+\psi_0)^\ell t}{\epsilon}$  be sufficiently large such that  $\mu_n[b, \infty) \leq \epsilon$  for each  $1 \leq n < M$ . For each  $n \geq M$  it follows from (6.1) that

$$\mu_n[b, \infty) = \mathbb{P}\{S_{N_n^\omega}^{A_n^\omega} \geq b\} \leq \frac{E[S_{N_n^\omega}^{A_n^\omega}]}{b} \leq \epsilon$$

which shows that  $\{\mu_n\}_{n=1}^\infty$  is tight.

*Proof of Assertion (b):* For each  $n \geq 1$  let  $W_n \sim \text{Pois}(t(1 - \rho_{A_n^\omega}))$  (where  $\rho_{A_n^\omega}$  is defined in section 3.1). From Theorem 2.3 in [KR], it follows that for each  $n \geq 1$  there exist a sequence of i.i.d. random variables  $\eta_{n,1}, \eta_{n,2}, \dots$ , independent of  $W_n$ , such that  $\mathbb{P}\{\eta_{n,1} \in \{1, \dots, \lfloor \frac{n}{r} \rfloor\}\} = 1$  and for  $Z_n = \sum_{j=1}^{W_n} \eta_{n,j}$ ,

$$(6.2) \quad \lim_{n \rightarrow \infty} d_{TV}(\mu_n, \mathcal{L}(Z_n)) = 0.$$

For each  $n \geq 1$  let  $\nu_n \in \mathcal{M}(\mathbb{N})$  be the distribution of  $Z_n$ . From (6.2) and since  $\{\mu_n\}_{n=1}^\infty$  is tight, it follows that  $\{\nu_n\}_{n=1}^\infty$  is also tight.

For each  $n \geq 1$  let  $\theta_n$  be the distribution of  $\eta_{n,1}, \eta_{n,2}, \dots$ . We shall now show that  $\{\theta_n\}_{n=1}^\infty$  is tight. Assume by contradiction that  $\{\theta_n\}_{n=1}^\infty$  is not tight, then there exist  $\epsilon > 0$  such that for each  $k \geq 1$  there exist  $n_k \geq 1$  with  $\theta_{n_k}[k, \infty) > \epsilon$ . From Theorem 2.3 in [KR] it follows that  $\sup_{n \geq 1} \rho_{A_n^\omega} < 1$ , so there exists  $\delta > 0$  such that

$$\inf_{n \geq 1} \mathbb{P}\{W_n > 0\} = \inf_{n \geq 1} (1 - \exp(-t(1 - \rho_{A_n^\omega}))) > \delta,$$

and so for each  $k \geq 1$ ,

$$\begin{aligned} \nu_{n_k}[k, \infty) &= \mathbb{P}\{Z_{n_k} \geq k\} \geq \mathbb{P}(\{W_{n_k} > 0\} \cap \{\eta_{n_k,1} \geq k\}) = \\ &= \mathbb{P}\{W_{n_k} > 0\} \cdot \theta_{n_k}[k, \infty) > \delta \cdot \epsilon \end{aligned}$$

which is a contradiction to the tightness of  $\{\nu_n\}_{n=1}^\infty$ , and so  $\{\theta_n\}_{n=1}^\infty$  must be tight. Let  $\mu$  be a probability distribution on  $\mathbb{R}$  such that  $\mu_{n_k} \xrightarrow{d} \mu$  as  $k \rightarrow \infty$ , for some increasing sequence  $\{n_k\}_{k=1}^\infty \subset \mathbb{N}_+$ . From (6.2) it follows that also  $\nu_{n_k} \xrightarrow{d} \mu$  as  $k \rightarrow \infty$ . For each  $k \geq 1$  set  $\tau_k = t(1 - \rho_{A_{n_k}^\omega})$ , then  $W_{n_k} \sim \text{Pois}(\tau_k)$  and

$$0 < t(1 - \sup_{n \geq 1} \rho_{A_n^\omega}) \leq \tau_k \leq t$$

for each  $k \geq 1$ . From this it follows, by moving to a sub-sequence without changing notation, that we can assume  $\tau_k \xrightarrow{k} \tau$  for some  $0 < \tau \leq t$ . Also, since  $\{\theta_n\}_{n=1}^\infty$  is tight and from Theorem 25.10 in [B1], by moving to a sub-sequence without changing notation, we can assume  $\theta_{n_k} \xrightarrow{d} \theta$  for some probability distribution  $\theta$  on  $\mathbb{R}$ . Since  $\theta_{n_k} \in \mathcal{M}(\mathbb{N})$  for each  $k \geq 1$ , it follows that  $\theta \in \mathcal{M}(\mathbb{N})$ .

It follows from Theorem 26.3 in [B1] and from (2.3), that for each  $x \in \mathbb{R}$

$$\begin{aligned} \varphi_\mu(x) &= \lim_{k \rightarrow \infty} \varphi_{\nu_{n_k}}(x) = \lim_{k \rightarrow \infty} \exp(\tau_k \cdot (\varphi_{\theta_{n_k}}(x) - 1)) = \\ &= \exp(\tau \cdot (\varphi_\theta(x) - 1)) = \varphi_{CP(\tau, \theta)}(x) \end{aligned}$$

which shows that  $\mu = CP(\tau, \theta)$ , and completes the proof.  $\square$

## 7. PROOF OF THEOREM 10

*Proof of Assertion (a):* Let  $g_0, \dots, g_{n-1} \in G$ , then since  $\mathbb{P}$  is  $T$ -invariant,

$$\begin{aligned} \mathbb{P}_0(T^{-1}[g_0, \dots, g_{n-1}]) &= \mathbb{P}(\Phi^{-1}\{\omega : \omega_{j+1} = g_j \text{ for each } 0 \leq j < n\}) = \\ &= \mathbb{P}\left(\bigcap_{j=0}^{n-1} \{\omega : \omega_{j+1} + \dots + \omega_{j+N} = g_j\}\right) = \mathbb{P}\left(T^{-1}\left(\bigcap_{j=0}^{n-1} \{\omega : \omega_j + \dots + \omega_{j+N-1} = g_j\}\right)\right) = \\ &= \mathbb{P}\left(\bigcap_{j=0}^{n-1} \{\omega : \omega_j + \dots + \omega_{j+N-1} = g_j\}\right) = \mathbb{P}(\Phi^{-1}[g_0, \dots, g_{n-1}]) = \mathbb{P}_0[g_0, \dots, g_{n-1}]. \end{aligned}$$

Since  $\mathcal{F}$  is generated by the  $\pi$ -system of all cylinders, it follows from the  $\pi - \lambda$  theorem, that  $\mathbb{P}_0$  is  $T$ -invariant.  $\square$

For the proof of Assertion (b) the following lemma will be needed.

**Lemma 14.** *Let  $l \in \mathbb{N}$  and  $C \geq 0$  be such that*

$$(7.1) \quad |\mathbb{P}_0(A \cap T^{-n-l}B) - \mathbb{P}_0(A)\mathbb{P}_0(B)| \leq C\mathbb{P}_0(A)\mathbb{P}_0(B)$$

*for each  $n$ -cylinder  $A \in \mathcal{F}_{\{0, \dots, n-1\}}$  and cylinder  $B \in \mathcal{F}$ . Then*

$$|\mathbb{P}_0(E \cap T^{-n-l}F) - \mathbb{P}_0(E)\mathbb{P}_0(F)| \leq C\mathbb{P}_0(E)\mathbb{P}_0(F)$$

*for each  $E \in \mathcal{F}_{\{0, \dots, n-1\}}$  and  $F \in \mathcal{F}$ .*

*Proof:* Since  $\mathcal{F}$  is generated by cylinders the result follows.  $\square$



*Proof of Assertion (b):* Let  $n, k \geq 1$ ,  $g_0, \dots, g_{n+k-1} \in G$ ,  $A = [g_0, \dots, g_{n-1}]$ ,  $B = [g_n, \dots, g_{n+k-1}]$  and  $\lambda = \min\{p_g : g \in G\}$ . We shall first show by induction on  $m$  that for each  $0 \leq m < N$

$$(7.2) \quad \mathbb{P}_0[g_0, \dots, g_{n-N}] \leq \lambda^{-m} \mathbb{P}_0[g_0, \dots, g_{n+m-N}].$$

For  $m = 0$ , (7.2) is obvious. Assume (7.2) is true for some  $0 \leq m < N - 1$ . If  $n + m < N - 1$  then  $[g_0, \dots, g_{n+m+1-N}] = \Omega$  and (7.2) is obvious, hence we can assume that  $n + m \geq N - 1$ . Set

$$\mathcal{E} = \{R \in \mathcal{F} : R = [x_0, \dots, x_{n+m-1}] \text{ and } x_j + \dots + x_{j+N-1} = g_j \text{ for } 0 \leq j \leq n+m-N\}$$

then

$$\Phi^{-1}[g_0, \dots, g_{n+m-N}] = \bigcup_{R \in \mathcal{E}} R.$$

For each  $[x_0, \dots, x_{n+m-1}] = R \in \mathcal{E}$  set

$$y_R = g_{n+m+1-N} - x_{n+m-1} - \dots - x_{n+m+1-N}$$

and  $Q_R = [x_0, \dots, x_{n+m-1}, y_R]$ . Then  $Q_R \subset \Phi^{-1}[g_0, \dots, g_{n+m+1-N}]$ , and so

$$\begin{aligned} \mathbb{P}_0[g_0, \dots, g_{n-N}] &\stackrel{\text{i.h.}}{\leq} \lambda^{-m} \mathbb{P}_0[g_0, \dots, g_{n+m-N}] = \lambda^{-m} \mathbb{P}(\Phi^{-1}[g_0, \dots, g_{n+m-N}]) = \\ &= \lambda^{-m} \sum_{R \in \mathcal{E}} \mathbb{P}(R) \leq \lambda^{-m-1} \sum_{R \in \mathcal{E}} \mathbb{P}(Q_R) = \lambda^{-m-1} \mathbb{P}\left(\bigcup_{R \in \mathcal{E}} Q_R\right) \leq \\ &\leq \lambda^{-m-1} \mathbb{P}(\Phi^{-1}[g_0, \dots, g_{n+m+1-N}]) = \lambda^{-m-1} \mathbb{P}_0[g_0, \dots, g_{n+m+1-N}] \end{aligned}$$

and the induction is complete.

Let  $0 \leq l < N - 1$ , then from (7.2) with  $m = N - 1$ ,

$$\begin{aligned} |\mathbb{P}_0(A \cap T^{-n-l}B) - \mathbb{P}_0(A)\mathbb{P}_0(B)| &\leq \mathbb{P}_0([g_0, \dots, g_{n-N}] \cap T^{-n-l}B) + \mathbb{P}_0(A)\mathbb{P}_0(B) = \\ &= \mathbb{P}_0[g_0, \dots, g_{n-N}]\mathbb{P}_0(B) + \mathbb{P}_0(A)\mathbb{P}_0(B) \leq (1 + \lambda^{-N+1})\mathbb{P}_0(A)\mathbb{P}_0(B) \end{aligned}$$

and for  $l \geq N - 1$ ,

$$\mathbb{P}_0(A \cap T^{-n-l}B) = \mathbb{P}_0(A)\mathbb{P}_0(B).$$

This together with Lemma 14 shows that  $\mathbb{P}_0$  is  $\psi$ -mixing.  $\square$

*Proof of Assertion (c):* Let  $h \in G$  be such that  $p_h > p_g$  for all  $g \in G \setminus \{h\}$ . The following notation will be needed. For  $m \in \mathbb{N}$  and  $g \in G$  set

$$m \cdot g := \underbrace{g + \dots + g}_{m \text{ times}}$$

Let  $G^*$  be the set of finite words over  $G$  (when  $G$  is thought of as an alphabet). As before, for each  $u, w \in G^*$  and  $k \geq 0$  let  $u \cdot w \in G^*$  be the concatenation of  $u$  and  $w$ , and let  $w^k \in G^*$  be the concatenation of  $w$  with itself  $k$  times.

Let  $r \in G \setminus \{h\}$  be such that  $p_r \geq p_g$  for all  $g \in G \setminus \{h\}$ , and set  $s = (N-1) \cdot h + r$ . We will prove (c) by showing that the limit  $\lim_n \mathbb{P}_0\{[s^{n+1}] \mid [s^n]\}$  does not exist. Set

$$\mathcal{E} = \{R \in \mathcal{F} : R = [g_0, \dots, g_{N-1}] \text{ and } g_0 + \dots + g_{N-1} = s\}$$

and for each  $0 \leq j < N$  set  $H_j = [h^j \cdot r \cdot h^{N-1-j}]$ , then since  $G$  is abelian  $H_0, \dots, H_{N-1} \in \mathcal{E}$ . For each  $n \geq 1$ , let  $a_n, b_n \in \mathbb{N}$  be such that  $n+N-1 = a_n \cdot N + b_n$  and  $0 \leq b_n < N$ .

We shall now show by induction on  $n \geq 1$  that  $\Phi^{-1}[s^n] = \bigcup_{R \in \mathcal{E}} R^{(n+N-1)/N}$ . For  $n = 1$  this follows directly from the definition of  $\mathcal{E}$ . Let  $n \geq 1$  and assume we know that  $\Phi^{-1}[s^n] = \bigcup_{R \in \mathcal{E}} R^{(n+N-1)/N}$ , then

$$(7.3) \quad \begin{aligned} \Phi^{-1}[s^{n+1}] &= \Phi^{-1}[s^n] \cap \Phi^{-1}\{\omega_n = s\} = \\ &= \left( \bigcup_{R \in \mathcal{E}} R^{(n+N-1)/N} \right) \cap \Phi^{-1}\{\omega_n = s\} = \bigcup_{R \in \mathcal{E}} (R^{(n+N-1)/N} \cap \Phi^{-1}\{\omega_n = s\}). \end{aligned}$$

Let  $[g_0, \dots, g_{N-1}] = R \in \mathcal{E}$ , then since  $G$  is abelian it follows for  $\omega \in R^{(n+N-1)/N}$  that

$$\omega_{n-1} + \dots + \omega_{n+N-2} = g_{b_n} + \dots + g_{N-1} + g_0 + \dots + g_{b_n-1} = s$$

and for  $\omega \in \Phi^{-1}\{\omega_n = s\}$ ,

$$\omega_n + \dots + \omega_{n+N-1} = s.$$

Hence, for  $\omega \in R^{(n+N-1)/N} \cap \Phi^{-1}\{\omega_n = s\}$ ,

$$\omega_{n-1} + \dots + \omega_{n+N-2} = s = \omega_n + \dots + \omega_{n+N-1}$$

which shows that  $\omega_{n-1} = \omega_{n+N-1}$ , and so  $\omega \in R^{(n+N)/N}$ . This shows that  $R^{(n+N-1)/N} \cap \Phi^{-1}\{\omega_n = s\} \subset R^{(n+N)/N}$ . On the other hand, if  $\omega \in R^{(n+N)/N}$  then

$$\omega_n + \dots + \omega_{n+N-1} = g_{b_{n+1}} + \dots + g_{N-1} + g_0 + \dots + g_{b_{n+1}-1} = s$$

so  $\omega \in R^{(n+N-1)/N} \cap \Phi^{-1}\{\omega_n = s\}$ , which shows that

$$R^{(n+N-1)/N} \cap \Phi^{-1}\{\omega_n = s\} = R^{(n+N)/N}.$$

Now from (7.3) it follows that  $\Phi^{-1}[s^{n+1}] = \bigcup_{R \in \mathcal{E}} R^{(n+N)/N}$  and the induction is complete.

Let  $n \geq 1$ , then

$$(7.4) \quad \begin{aligned} \mathbb{P}_0[s^n] &= \mathbb{P}(\Phi^{-1}[s^n]) = \mathbb{P}\left(\bigcup_{R \in \mathcal{E}} R^{(n+N-1)/N}\right) = \sum_{R \in \mathcal{E}} \mathbb{P}(R^{(n+N-1)/N}) = \\ &= \sum_{[g_0, \dots, g_{N-1}] \in \mathcal{E}} (p_{g_0} \cdot \dots \cdot p_{g_{N-1}})^{a_n} \cdot p_{g_0} \cdot \dots \cdot p_{g_{b_n-1}} \end{aligned}$$

Set  $\mathcal{Q} = \{H_0, \dots, H_{N-1}\}$  and let  $[g_0, \dots, g_{N-1}] \in \mathcal{E} \setminus \mathcal{Q}$ , we shall now show that  $\frac{p_{g_0} \cdots p_{g_{N-1}}}{p_h^{N-1} \cdot p_r} < 1$ . Let  $t \in G \setminus \{r\}$ , then

$$(N-1)h + t \neq (N-1)h + r = s$$

and so  $[g_0, \dots, g_{N-1}] \neq [h^j \cdot t \cdot h^{N-1-j}]$  for each  $0 \leq j < N$ . Since  $[g_0, \dots, g_{N-1}] \notin \mathcal{Q}$ , it follows that there exist  $0 \leq i < j < N$  with  $g_i, g_j \neq h$ . Since  $p_h > p_g$  and  $p_r \geq p_g$  for each  $g \in G \setminus \{h\}$ , it follows that

$$(7.5) \quad \frac{p_{g_0} \cdots p_{g_{N-1}}}{p_h^{N-1} \cdot p_r} \leq \frac{p_{g_i} \cdot p_{g_j}}{p_h \cdot p_r} < 1.$$

From  $\mathcal{Q} \subset \mathcal{E}$ , (7.4) and (7.5) we get that

$$\begin{aligned} 1 &\leq \limsup_n \frac{\mathbb{P}_0[s^n]}{\sum_{R \in \mathcal{Q}} \mathbb{P}(R^{(n+N-1)/N})} \leq \\ &\leq 1 + \limsup_n \sum_{[g_0, \dots, g_{N-1}] \in \mathcal{E} \setminus \mathcal{Q}} \frac{(p_{g_0} \cdots p_{g_{N-1}})^{a_n}}{(p_h^{N-1} \cdot p_r)^{a_n+1}} = 1, \end{aligned}$$

and so

$$(7.6) \quad \lim_n \mathbb{P}_0\{[s^{n+1}] \mid [s^n]\} \cdot \left( \frac{\sum_{R \in \mathcal{Q}} \mathbb{P}(R^{(n+N)/N})}{\sum_{R \in \mathcal{Q}} \mathbb{P}(R^{(n+N-1)/N})} \right)^{-1} = 1.$$

Let  $n \geq 1$ , then if  $b_n = 0$ ,

$$(7.7) \quad \begin{aligned} \frac{\sum_{R \in \mathcal{Q}} \mathbb{P}(R^{(n+N)/N})}{\sum_{R \in \mathcal{Q}} \mathbb{P}(R^{(n+N-1)/N})} &= \\ &= \frac{(p_h^{N-1} \cdot p_r)^{a_n} \cdot p_r + (N-1)(p_h^{N-1} \cdot p_r)^{a_n} \cdot p_h}{N \cdot (p_h^{N-1} \cdot p_r)^{a_n}} = \frac{p_r + (N-1)p_h}{N} \end{aligned}$$

and if  $b_n = N-1$  then

$$(7.8) \quad \begin{aligned} \frac{\sum_{R \in \mathcal{Q}} \mathbb{P}(R^{(n+N)/N})}{\sum_{R \in \mathcal{Q}} \mathbb{P}(R^{(n+N-1)/N})} &= \\ &= \frac{N \cdot (p_h^{N-1} \cdot p_r)^{a_n+1}}{(N-1)(p_h^{N-1} \cdot p_r)^{a_n} \cdot p_h^{N-2} \cdot p_r + (p_h^{N-1} \cdot p_r)^{a_n} \cdot p_h^{N-1}} = \frac{N \cdot p_h \cdot p_r}{(N-1) \cdot p_r + p_h}. \end{aligned}$$

Now if

$$\frac{p_r + (N-1)p_h}{N} = \frac{N \cdot p_h \cdot p_r}{(N-1) \cdot p_r + p_h},$$

a direct computation shows that  $p_h = p_r$ , so according to our assumptions it must hold that

$$(7.9) \quad \frac{p_r + (N-1)p_h}{N} \neq \frac{N \cdot p_h \cdot p_r}{(N-1) \cdot p_r + p_h}.$$

From (7.6), (7.7), (7.8) and (7.9) it follows that  $\lim_n \mathbb{P}_0\{[s^{n+1}] \mid [s^n]\}$  does not exist, and the theorem is proved.  $\square$

## 8. PROOF OF THEOREM 12

For  $b_0, \dots, b_{n-1} \in \{0, 1\}$  we write

$$[b_0, \dots, b_{n-1}]_0 = \{\omega \in \Omega_0 : \omega_j = b_j \text{ for each } 0 \leq j < n\}.$$

*Proof of Assertion (a):* Let  $b_0, \dots, b_{n-1} \in \{0, 1\}$ , then since  $\mathbb{P}$  is  $T$ -invariant,

$$\begin{aligned} \mathbb{P}_0(T_0^{-1}[b_0, \dots, b_{n-1}]_0) &= \mathbb{P}(\Theta^{-1}\{\omega \in \Omega_0 : \omega_{j+1} = b_j \text{ for each } 0 \leq j < n\}) = \\ &= \mathbb{P}\{\omega \in \Omega : \theta(\omega_{j+1}, \omega_{j+2}) = b_j \text{ for each } 0 \leq j < n\} = \\ &= \mathbb{P}(T^{-1}\{\omega \in \Omega : \theta(\omega_j, \omega_{j+1}) = b_j \text{ for each } 0 \leq j < n\}) = \\ &= \mathbb{P}\{\omega \in \Omega : \theta(\omega_j, \omega_{j+1}) = b_j \text{ for each } 0 \leq j < n\} = \\ &= \mathbb{P}(\Theta^{-1}\{\omega \in \Omega_0 : \omega_j = b_j \text{ for each } 0 \leq j < n\}) = \mathbb{P}_0[b_0, \dots, b_{n-1}]_0. \end{aligned}$$

Since  $\mathcal{F}_0$  is generated by the  $\pi$ -system of all cylinders, it follows that  $\mathbb{P}_0$  is  $T_0$ -invariant.  $\square$

For the proof of Assertion (b) we shall need the following lemma.

**Lemma 15.** *Let  $b_0, \dots, b_{n-1} \in \{0, 1\}$ ,  $D = \Theta^{-1}[b_0, \dots, b_{n-1}]_0$  and integers  $1 \leq r < k \leq s$  be given, then*

$$\mathbb{P}(\{\omega_0 = s\} \cap D) \leq \mathbb{P}(\{\omega_0 = r\} \cap D) + \mathbb{P}(\{\omega_0 = k\} \cap D).$$

*Proof:* If  $b_j = 1$  for each  $0 \leq j < n$  then

$$\begin{aligned} \mathbb{P}(\{\omega_0 = s\} \cap D) &= \mathbb{P}\{\omega_j = s + j : \text{for each } 0 \leq j \leq n\} = \prod_{j=0}^n 2^{-s-j} \leq \\ &\leq \prod_{j=0}^n 2^{-k-j} = \mathbb{P}\{\omega_j = k + j : \text{for each } 0 \leq j \leq n\} = \mathbb{P}(\{\omega_0 = k\} \cap D). \end{aligned}$$

Hence, we can assume that there exist  $0 \leq j_0 < n$  such that  $b_{j_0} = 0$  and  $b_j = 1$  for each  $0 \leq j < j_0$ . Set

$$D' = \{\omega \in \Omega : \theta(\omega_j, \omega_{j+1}) = b_j \text{ for each } j_0 + 1 \leq j < n\}.$$

Then it follows that

$$\begin{aligned} \mathbb{P}(\{\omega_0 = r\} \cap D) + \mathbb{P}(\{\omega_0 = k\} \cap D) &= \\ &= \mathbb{P}(\{\omega_j = r + j : \text{for each } 0 \leq j \leq j_0\} \cap \{\omega_{j_0+1} \neq r + j_0 + 1\} \cap D') + \\ &\quad + \mathbb{P}(\{\omega_j = k + j : \text{for each } 0 \leq j \leq j_0\} \cap \{\omega_{j_0+1} \neq k + j_0 + 1\} \cap D') \geq \\ &\geq \mathbb{P}\{\omega_j = s + j : \text{for each } 0 \leq j \leq j_0\} (\mathbb{P}(\{\omega_{j_0+1} \neq r + j_0 + 1\} \cap D') + \mathbb{P}(\{\omega_{j_0+1} \neq k + j_0 + 1\} \cap D')) \geq \\ &\geq \mathbb{P}\{\omega_j = s + j : \text{for each } 0 \leq j \leq j_0\} \mathbb{P}(D') \geq \mathbb{P}(\{\omega_0 = s\} \cap D) \end{aligned}$$

and the lemma is proved.  $\square$

*Proof of Assertion (b):* Let  $n \geq 0$ ,  $l \geq 1$ , and  $b_0, \dots, b_{n+l} \in \{0, 1\}$ . Set

$$B_1 = \{\omega \in \Omega : \theta(\omega_j, \omega_{j+1}) = b_j \text{ for each } j \in \{0, \dots, n-1\}\}$$

and

$$B_2 = \{\omega \in \Omega : \theta(\omega_j, \omega_{j+1}) = b_j \text{ for each } j \in \{n+1, \dots, n+l\}\}.$$

Then  $B_1$  and  $B_2$  are independent events, and so

$$\begin{aligned} (8.1) \quad & \mathbb{P}_0([b_0, \dots, b_{n-1}, 0]_0 \cap T_0^{-(n+1)}[b_{n+1}, \dots, b_{n+l}]_0) \leq \mathbb{P}(B_1 \cap B_2) = \\ & = \mathbb{P}(B_1)\mathbb{P}(B_2) = \mathbb{P}(B_2) \sum_{k=1}^{\infty} \mathbb{P}(B_1 \cap \{\omega_n = k\}) \mathbb{P}\{\omega_{n+1} \neq k+1\} (\mathbb{P}\{\omega_{n+1} \neq k+1\})^{-1} \leq \\ & \leq \mathbb{P}(B_2) \sum_{k=1}^{\infty} \mathbb{P}(B_1 \cap \{\omega_n = k\} \cap \{\omega_{n+1} \neq k+1\}) \cdot 2 = 2\mathbb{P}(B_2) \mathbb{P}(B_1 \cap \{\theta(\omega_n, \omega_{n+1}) = 0\}) = \\ & = 2\mathbb{P}_0[b_0, \dots, b_{n-1}, 0]_0 \mathbb{P}_0[b_{n+1}, \dots, b_{n+l}]_0. \end{aligned}$$

In a similar manner it can be shown that

$$(8.2) \quad \mathbb{P}_0([b_0, \dots, b_n]_0 \cap T_0^{-(n+1)}[0, b_{n+2}, \dots, b_{n+l}]_0) \leq 2\mathbb{P}_0[b_0, \dots, b_n]_0 \mathbb{P}_0[0, b_{n+2}, \dots, b_{n+l}]_0.$$

Now set

$$B_3 = \{\omega \in \Omega : \theta(\omega_j, \omega_{j+1}) = b_j \text{ for each } j \in \{n+2, \dots, n+l\}\}.$$

Then

$$\begin{aligned} (8.3) \quad & \mathbb{P}_0([b_0, \dots, b_{n-1}, 1]_0 \cap T_0^{-(n+1)}[1, b_{n+2}, \dots, b_{n+l}]_0) = \\ & = \sum_{k=1}^{\infty} \mathbb{P}(B_1 \cap \{\omega_n = k\} \cap \{\omega_{n+1} = k+1\} \cap \{\omega_{n+2} = k+2\} \cap B_3) = \\ & = \sum_{k=1}^{\infty} \mathbb{P}(B_1 \cap \{\omega_n = k\} \cap \{\omega_{n+1} = k+1\}) \mathbb{P}(\{\omega_{n+2} = k+2\} \cap B_3) \stackrel{\text{lemma 15}}{\leq} \\ & \leq (\mathbb{P}(\{\omega_{n+2} = 3\} \cap B_3) + \mathbb{P}(\{\omega_{n+2} = 2\} \cap B_3)) \sum_{k=1}^{\infty} \mathbb{P}(B_1 \cap \{\omega_n = k\} \cap \{\omega_{n+1} = k+1\}) = \\ & = (4\mathbb{P}(\{\omega_{n+1} = 2\} \cap \{\omega_{n+2} = 3\} \cap B_3) + 2\mathbb{P}(\{\omega_{n+1} = 1\} \cap \{\omega_{n+2} = 2\} \cap B_3)) \mathbb{P}_0[b_0, \dots, b_{n-1}, 1]_0 \leq \\ & \leq 6\mathbb{P}_0[b_0, \dots, b_{n-1}, 1]_0 \mathbb{P}_0[1, b_{n+2}, \dots, b_{n+l}]_0. \end{aligned}$$

From (8.1), (8.2), (8.3) and Lemma 14 it follows that

$$|\mathbb{P}_0(E \cap T^{-n-1}F) - \mathbb{P}_0(E)\mathbb{P}_0(F)| \leq 7\mathbb{P}_0(E)\mathbb{P}_0(F)$$

for each  $E \in \mathcal{F}_{0,\{0,\dots,n\}}$  and  $F \in \mathcal{F}_0$ .

Let  $s > n + 1$ , then

$$\mathbb{P}_0([b_0, \dots, b_n]_0 \cap T_0^{-s}[b_{n+1}, \dots, b_{n+l}]_0) = \mathbb{P}_0[b_0, \dots, b_n]_0 \mathbb{P}_0[b_{n+1}, \dots, b_{n+l}]_0$$

so by Lemma 14,

$$\mathbb{P}_0(E \cap T^{-s}F) = \mathbb{P}_0(E)\mathbb{P}_0(F)$$

for each  $E \in \mathcal{F}_{0,\{0,\dots,n\}}$  and  $F \in \mathcal{F}_0$ . This shows that  $\mathbb{P}_0$  is  $\psi$ -mixing.  $\square$

*Proof of Assertion (c):* For each integer  $l \in \mathbb{N}$  set  $f(l) = 2^{-l}$ . Let  $n \geq 1$ , then

$$\begin{aligned} \mathbb{P}_0[1^n]_0 &= \sum_{k=1}^{\infty} \mathbb{P}[k, \dots, k+n] = \sum_{k=1}^{\infty} \prod_{j=k}^{k+n} 2^{-j} = \\ &= \sum_{k=1}^{\infty} f\left(\sum_{j=k}^{k+n} j\right) = \sum_{k=1}^{\infty} f\left(\frac{(n+1)(2k+n)}{2}\right) = \\ &= f\left(\frac{(n+1)n}{2}\right) \sum_{k=1}^{\infty} (2^{-(n+1)})^k = f\left(\frac{(n+1)n}{2}\right) \cdot 2^{-(n+1)} \cdot \frac{1}{1-2^{-(n+1)}}. \end{aligned}$$

Hence,

$$\begin{aligned} \mathbb{P}_0\{[1^{n+1}] \mid [1^n]\} &= \frac{f\left(\frac{(n+2)(n+1)}{2}\right) \cdot 2^{-(n+2)} \cdot \frac{1}{1-2^{-(n+2)}}}{f\left(\frac{(n+1)n}{2}\right) \cdot 2^{-(n+1)} \cdot \frac{1}{1-2^{-(n+1)}}} \leq \\ &\leq f\left(\frac{(n+2)(n+1)}{2} - \frac{(n+1)n}{2}\right) = 2^{-(n+1)} \xrightarrow{n \rightarrow \infty} 0 \end{aligned}$$

and the theorem is proved.  $\square$

## REFERENCES

- [AV1] Abadi, M., Vergne, N.: Poisson approximation for search of rare words in DNA sequences. ALEA Lat. Am. J. Probab. Math. Stat. 4, 223–244 (2008).
- [AV2] Abadi, M., Vergne, N.: Sharp errors for point-wise Poisson approximations in mixing processes. Nonlinearity 21, 2871–2885 (2008).
- [BHJ] A.D. Barbour, L. Holst and S. Janson, Poisson Approximation, Oxford Univ. Press, Oxford (1992).
- [B1] Billingsley, Patrick. Probability and Measure. Wiley Series in Probability and Mathematical Statistics, (1995).
- [B2] R. Bowen, Equilibrium States and the Ergodic Theory of Anosov Diffeomorphisms, Lecture Notes in Math. 470, Springer-Verlag, Berlin, 1975.
- [FFT] Freitas, A.C.M., Freitas, J.M., Todd, M.: The compound Poisson limit ruling periodic extreme behavior of non-uniformly hyperbolic dynamic. Commun. Math. Phys. 321, 483–527 (2013).
- [H] Heinrich, L.: Mixing properties and central limit theorem for a class of non-identical piecewise monotonic C2-transformations. Mathematische Nachricht. 181, 185–214 (1996).

- [HV] Haydn, N., Vaienti, S.: The compound Poisson distribution and return times in dynamical systems. *Probab. Theory Relat. Fields* 144, 517–542 (2009).
- [K] Kifer, Y.: Nonconventional Poisson limit theorems. *Israel J. Math.* 195, 373–392 (2013).
- [KR] Y. Kifer and A. Rapaport. Poisson and compound Poisson approximations in conventional and nonconventional setups. *Probability Theory and Related Fields*, to appear (2014).
- [S] O. Sarig, Lecture notes on thermodynamic formalism for topological Markov shifts, Penn State, preprint 2009.
- [W] Walters, P.: Some results on the classification of non-invertible measure preserving transformations. *Lecture Notes in Math.* 318, Berlin: Springer, 1973, pp. 266–276.